Strange Loops and Tangled Hierarchies

One of my first assignments as a tax lawyer involved the takeover battle between Kennecott and Curtiss-Wright. This was one of the first occasions for the use of the "Pac-Man" defense, where the target launches a counterattack by buying shares of its purported acquirer.[1] The result, of course, is that both companies send up owning shares of each other.

As I started looking at the tax consequences of such a situation, it became apparent that the tax law was ill-equipped to deal with corporate cross-ownership. Even sorting out the economics of these arrangements was no trivial matter. And this was a relatively simple case involving just two corporations.

At that time Japan was looked to a model of the future of capitalism. A feature of the Japanese economy is the ownership of significant stakes by corporations in other corporations, forming a complex web of alliances. I started wondering, how do they figure out who ultimately owns what?

This paper is an attempt to analyze, in a systematic way, the consequences of corporate cross-ownership, no matter how many corporations are involved, and no matter how complex the linkages are among them. Doing so required developing a notation that could represent a messy corporate chart in a way that was susceptible to analysis. This can be done my means of a simple table that shows the percentage stock ownership of each corporation in each other corporation. Such a table can be viewed mathematically as a matrix, which opens up the entire field of matrix algebra as a source of tools to answer the questions posed by these arrangements.

I settled on the term "strange loop" for the situation where two corporations own stock in each other. In the years since this article

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[1]   *See* JAMES B. STEWART, THE PARTNERS 258 (1983).

was published,[2] that term never caught on, the term "hook stock" being used instead. As the paper points out, the terms "strange loop" and "tangled hierarchy" were borrowed from Douglas Hofstadter, who uses them to describe a variety of forms of indirect self-reference in logic and language. I alerted Professor Hofstadter to my appropriation of his terms in the field of corporate tax law, and to my relief he was perfectly happy with that.

This paper does not shy away from the use of equations and mathematical formulas. I appreciate that this will put off many read-ers; indeed, one such reader, a noted tax practitioner, took the trouble to write to me in order to express how offended he was by my use of math. Of course no offense was intended, and I hope that math-phobic readers will appreciate that the very task I set for myself—to create a formal theory of strange loops—made the use of mathemati-cal language unavoidable.

Much of the paper consists of playful imaginings of how the tax law might apply to bizarre cross-ownership arrangements that are highly unlikely to arise in real life. But the Parts that discuss the crea-tion and unwinding of strange loops deal with actual arrangements, such as those involving McDermott and May Department Stores, that have been the subject of IRS guidance and case law.

The analysis of the creation and unwinding of strange loops re-mains incomplete. The paper only deals with two-corporation cases, and in an *ad hoc* manner. What is needed is a more general way of representing transactions that change the number and strength of links of cross-ownership, that can provide insights into the proper computation of gain and loss, and the measurement of any deemed distributions, that result from adjustments to cross-ownership ar-rangements. But that will need to be the subject of a future paper.

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[2]   Stephen B. Land, *Strange Loops and Tangled Hierarchies*, 52 TAX L. REV. 45 (1996).

Contents

I. INTRODUCTION

A corporation is a marvelous fiction. As a juridical person, it can own property, including shares of other corporations; it also can be owned, including by other corporations. Subsidiaries can be nested to any depth. The costs of forming a subsidiary are low, and the advantages are numerous: They can limit liability, isolate regulated operations and save taxes. Not surprisingly, the corporate charts of sizable business enterprises are frequently quite intricate. Yet, however structured, the typical corporate "tree" resembles an ordinary tree: From a single trunk emerge successively smaller branches. There is an orderly chain connecting the underlying assets (that is, assets other than shares of corporations in the tree) and their ultimate owners.

Nothing in corporate or tax law requires this to be the case, and occasionally a corporation acquires shares in another corporation that in turn owns part or all of it. When this happens, a "strange loop" is created: A corporation owns, indirectly, part of itself, and the hierarchy of ownership becomes tangled.[3] The Code contains some provisions, most notably Section 304, that specifically address the formation of strange loops. Yet, almost all other aspects of the tax law ignore their possibility, creating weird opportunities and pitfalls, including taxable gains and losses from sales of self-owned interests, and infinite rounds of taxation on dividends that cycle back and forth.

Strange loops have a paradoxical flavor. The unrestricted freedom of corporations to own shares in other corporations, including themselves, resembles set theory's law of comprehension, which implies that any set can be a member of any set, including itself.[4] The law of comprehension leads to Russell's paradox when a set is defined to

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[3]   The terms "strange loop" and "tangled hierarchy" come from DOUGLAS R. HOFSTADTER, GÖDEL, ESCHER, BACH: AN ETERNAL GOLDEN BRAID (1979), an imaginative tour de force built on the theme of indirect self-reference.

[4]   *See* WILLARD V.O. QUINE, SET THEORY AND ITS LOGIC 35–38 (rev. ed. 1971).

include every set that is not a member of itself.[5] This paradox has led to limits on the law of comprehension, the most well-known being Russell's ramified theory of types, in which each set is assigned a type level, and is permitted to have other sets as elements only if they are of a lower type.[6] Imagine corporate cross ownership run amok. Is it always possible to state who ultimately owns what, or can there be some perversely configured arrangement that creates a potential for paradox? Is a corporate theory of types necessary?

A sample strange loop: Two corporations, C_1 and C_2, each have tax loss carryforwards and a single class of stock. C_1 owns an asset worth \$100, and C_2 owns an asset worth \$200. Individual I_1 owns 10% of the stock of C_1, and unrelated individual I_2 owns 10% of the stock of C_2. C_1 and C_2 each own 90% of the stock of each other. Figure 1 shows the corporate diagram.

FIGURE 1

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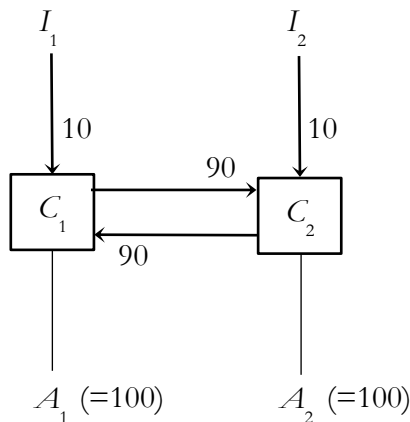
[5]   The paradox arises because the set of all objects that satisfies this condition is a member of itself if and only if it is not. Letter from Bertrand Russell to Gottlob Frege (June 16, 1902), in FROM FREGE TO GÖDEL: A SOURCE BOOK IN MATHEMATICAL LOGIC, 1879–1931, at 124–25 (Jean van Heijenoort, ed. 1967).

[6]   Bertrand Russell, *Mathematical Logic as Based on the Theory of Types*, 30 AM. J. MATH. 222–62 (1908).

I_1 sells all of her C_1 stock to unrelated individual I_3. Does this sale cause a 50% ownership change of C_1, triggering the Section 382 limitation on its loss carryforwards?[7] Are C_2's carryforwards limited? In either case, if the limitation applies, what is the value of the corporation's stock for purposes of computing the amount of the limitation?

I strongly urge the reader to think about this problem before reading past this section. The structure, though fanciful, provides a good test of one's intuitions about indirect self-ownership. Apart from the intricacies of Section 382, there are more basic questions to consider. If a corporation's stock is assumed to be worth the same as its assets, how much is I_1's stock worth? What is I_1's indirect percentage interest in asset A_1?

This example is simple enough to be solved with ad hoc methods.[8] It is worth asking, however, whether a general solution exists for a group of corporations with jumbled cross ownership, whether it is unique and how it can be found. To answer these questions in a precise way, Part III develops a formal theory of strange loops. Succeeding Parts apply this theory to specific areas of the tax law.

The formal theory is essentially a system of notation for describing chains of corporate ownership, including circular chains, that makes it easier to analyze direct and indirect ownership interests. The notation uses mathematical symbols, and the discussion below relies heavily on numerical formulas and examples. Sufficient detail is given so that mathematically inclined readers can apply the computational techniques presented here to any situation involving strange loops.

Readers who are content to take the computational techniques on faith can gloss over the math, and focus on the results. These results include a way of viewing indirectly self-owned interests as a distinct type of corporate asset, somewhat like treasury stock. Transactions in

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[7]   I.R.C. § 382 (a), (b), (g) (limiting the use of a corporation's net operating loss carryforwards following an ownership change of more than 50% during a three-year testing period).

[8]   *See infra* Part III.B (p. 185).

which corporations acquire or dispose of indirect interests in them-
selves deserve to be viewed differently from other types of
transactions in which corporations buy and sell assets. This article
provides such a viewpoint.

　　Strange loops, although rare, do crop up from time to time, most
recently as devices to enable corporations to dispose of interests in
appreciated property without a corporate tax. One example is the
highly publicized May Department Stores transaction, in which a
partnership acquired stock of one of its principal partners.[9] Another
is the exchange by McDermott International of its shares for shares
of its parent.[10] Every transaction covered by Section 304(a)(2), which
deals with purchases of parent shares by a subsidiary, creates a strange
loop. By and large, the tax law treats stock forming part of a strange
loop like any other asset, ignoring the indirect self-ownership inherent
in any strange loop. The failure to recognize the special features of
strange loopiness creates problems with the tax law that need to be
addressed, but only with a proper understanding of how strange loops
work.

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[9]   *See infra* notes 89–91 and accompanying text.

[10]   *See infra* note 93 and accompanying text.

II. INDIRECTLY SELF-OWNED STOCK

A. *Treasury Stock*

Treasury stock is the shortest possible strange loop. By and large, the tax law deals with this sort of strange loop quite intelligently, by ignoring it.[11] For purposes of the income tax, it makes no difference whether reacquired shares are retired, or whether shares sold by the issuer are newly issued. It was not always so. Regulations issued in 1934 provided that a corporation could recognize gain or loss from dealings in treasury stock,[12] contrary to a previous Board of Tax Appeals case.[13] The senseless distinction between treasury stock and newly issued stock disappeared in 1954 with the enactment of Section 1032.[14] Congress extended that section in 1984 to prevent a corporation from recognizing gain or loss on the lapse or repurchase of an

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[11] Even as a creature of corporate law, treasury stock may be an endangered species. Delaware did away with the concept years ago, and treasury shares now have the same legal status in Delaware as any other authorized but unissued shares. DEL. CORP. ANN. tit. 8, § 160 (1991). Other states, such as New York, still recognize treasury stock as such. N.Y. BUS. CORP. LAW § 515(b) (McKinney 1986).

[12] T.D. 4430, XIII-1 C.B. 36 (1934).

[13] Appeal of Simmons & Hammond Mfg. Co., 1 B.T.A. 803 (1925).

[14] Pub. L. No. 591, 83d Cong., 2d Sess. 91032, 68A Stat. 3, 303. The demise of this distinction is confirmed by the regulations under Section 1032. Treas. Reg. § 1.1032-l(a). For a further discussion of the background to Section 1032, see Elliott Manning, *The Issuer's Paper: Property or What? Zero Basis and Other Income Tax Mysteries*, 39 TAX L. REV. 159, 165–66 (1984). Occasionally, the Service forgets this point. In G.C.M. 39608 (Mar. 5, 1987), the Service concluded that if a subsidiary owned appreciated stock of its parent, any gain realized by the subsidiary upon a distribution of that stock to the parent could be deferred until the parent sold *those particular shares*. Regulations have been proposed since that would reverse this result, providing for immediate recognition of gain upon the distribution. Prop. Treas. Reg. § 1.1502-13(f)(4), 59 Fed. Reg. 18,011 (Apr. 15, 1994). It can be questioned, however, whether a transition from indirect to direct self-ownership of shares should be a taxable event at all.

option to buy its stock.[15] Section 1032 does not prevent a corporation from recognizing gain or loss on the issuance of an option, because it does not need to: there is no legal concept of a "treasury option," so the issuance of an option is always a newly created obligation, which is tax-free without regard to Section 1032, just like the issuance of a note.[16]

B.   *The Zero Basis Problem*

The Service occasionally treats stock, issued or not, as if it were an asset in the hands of its issuer, with nonsensical results. The most notorious example is the "zero basis" problem that is thought to arise when stock is issued in a carryover basis transaction. In Revenue Ruling 74-503,[17] corporation *X* transferred to corporation *Y* in a Section 351 exchange treasury shares of *X* that *X* had previously purchased from its shareholders. The ruling properly concluded that *X*'s cost of purchasing the treasury shares did not create a basis in these shares in the hands of the corporation, since the repurchase was, for tax purposes, equivalent to a redemption of the shares. Unfortunately, the Service decided that no basis meant zero basis, so under the carryover basis rules applicable to Section 351,[18] the *X* stock had a zero basis in *Y*'s hands, and the *Y* stock issued in exchange had a zero basis in *X*'s hands.

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[15]   I.R.C. § 1032(a).

[16]   See Douglas H. Walter, *"The Issuer's Own Stock"—Section 1032, Section 304 and Beyond*, 68 TAXES 906–07 (1990). The proceeds of the issuance of an option are nontaxable at that time even where the writer of the option is not the issuer of the underlying stock. Rev. Rul. 78-182, 1978-1 C.B. 265.

[17]   1974-2 C.B. 117.

[18]   I.R.C. §§ 358, 362.

Commentators unanimously have condemned the zero basis conclusion reached in Revenue Ruling 74-503.[19] Moreover, the adverse results can be avoided easily by careful planning: For example, the parent can sell its own stock to its subsidiary in a transaction that is tax-free to the parent under Section 1032, but gives the subsidiary a fair market value basis. The correct approach would be to provide that the basis of stock issued in a carryover basis transaction is its fair market value, so that a subsequent sale by the holder at that price does not result in a taxable gain that would be exempt under Section 1032 if realized by the issuer itself.

The rationale for this approach has been extensively discussed elsewhere and is not repeated here.[20] For purposes of this strange loops analysis, what is interesting is how the stock is treated after it is issued.[21] While all would agree that stock does not constitute "property" in the hands of its issuer, should it constitute property in the hands of its affiliate? Even in the starkest case, where a wholly-owned subsidiary holds stock of its parent, the Service treats the parent stock in the hands of the subsidiary as property, and its disposition can generate a taxable gain or loss.[22] Yet this "asset" of the subsidiary adds nothing to the parent's net worth: It just creates an indirect interest in itself. To the subsidiary, however, the parent stock represents partly an indirect interest in itself and partly an indirect interest in the parent's other assets. Unlike treasury stock which, if recognized at all under

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[19] E.g., Sheldon I. Banoff, *How IRS' New Zero-Basis Approach Will Affect Corporate Tax Planning*, 42 J. TAX'N 96 (1975); Manning, *supra* note 14, at 161–62; Walter, *supra* note 16, at 910.

[20] *See, e.g., id.* There is, perhaps, less agreement on the reasons why a fair market value basis is justified than on the conclusion itself. See Calvin H. Johnson, *The Legitimacy of Basis from a Corporation's Own Stock*, 9 AM. J. TAX POL'Y 155 (1991).

[21] *See* Peter C. Canellos, *Acquisition of Issuer Securities by a Controlled Entity: Peter Pan Seafood, May Department Stores, and McDermott*, 45 TAX LAW. 1, 7 (1991). Although it does not use the term, that article is all about strange loops, and, despite its brevity, anticipates the principal points made here.

[22] Rev. Rul. 70-305, 1970-1 C.B. 169, modified by Rev. Rul. 74-503, 1974-2 C.B. 117.

state law, has no legal consequences except perhaps some purely pro-cedural corporate formalities, the holding of parent stock by a subsidiary does make a difference. One way to see this difference is to look at the effects on creditors of the parent and the subsidiary.[23]

If, as in Revenue Ruling 74-503, a parent transfers its own stock to its subsidiary in exchange for more stock of the subsidiary, the parent has gained nothing but an indirect ownership interest in itself, which is worthless to its own creditors. From the point of view of the parent's creditors, parent stock held by a subsidiary does not add any value to the parent's investment in the subsidiary. To the subsidiary, however, the parent stock represents, in part, an interest in the parent's other assets. To that extent, from the point of view of the subsidiary's creditors, the stock of the parent is a valuable asset of the subsidiary. In effect, the subsidiary's creditors benefit at the expense of the parent's other shareholders, because these shareholders' interest in the parent can be diluted to satisfy the claims of the subsidiary's creditors. Here, the strange loop partly undermines the limited liability afforded by the parent-subsidiary relationship.

This analysis shows that indirectly self-owned stock has some ef-fect on the legal relationships among the parties that is lacking in the case of treasury stock. Whether indirectly self-owned stock should be treated as outstanding for tax purposes, however, depends on consid-erations that relate particularly to tax policy. For example, treating parent stock held by a wholly-owned subsidiary as outstanding for tax purposes could enable corporations to use Section 1032 to whipsaw the federal government. Consider a public corporation that plans a public offering of its stock in the foreseeable future. Each time the company's stock price reaches a new high, the corporation places a block of its stock in the hands of a special-purpose subsidiary, with the block having a fair market value basis that depends on the trading

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[23] I am indebted to Peter Canellos for this creditor-oriented approach to strange loops. *See* Canellos, *supra* note 21, at 2–5.

price of the parent's stock at that time. When the time comes to do the public offering, the subsidiary with the block of shares that has the highest basis sells its stock to the public, claiming a capital loss that can be used to offset capital gains elsewhere in the consolidated group.[24] If the parent's stock price at the time of the public offering is higher than the price reflected in the basis of the parent stock held by its subsidiaries, then the parent issues its own stock, avoiding any gain recognition under Section 1032. Any leftover strange loops can be eliminated when convenient by tax-free liquidations of the special-purpose subsidiaries under Section 332.

The Section 1032 whipsaw will not work if the parent stock in the hands of the subsidiary has a zero basis, which is perhaps the one good thing that can be said about the Service's conclusion in Revenue Ruling 74-503. Even this is faint praise, however, since the zero basis result can be avoided so easily by having the subsidiary purchase the parent shares for cash, or by having the parent contribute its own shares in a transaction that technically violates Section 351's control requirement.[25]

C. *Voting Rights*

Corporation law recognizes the unreality of treasury stock by re-fusing to permit an issuer to exercise voting rights on its treasury shares. For example, Delaware law provides that treasury stock "shall neither be entitled to vote nor be counted for quorum purposes."[26]

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[24] Although Treas. Reg. § 1.1502-20(a)(1) generally disallows a loss deduction on the sale of a subsidiary's stock, there is no comparable restriction on the sale by a subsidiary of stock of the common parent.

[25] For example, if an unrelated party held a small amount of straight nonvoting preferred stock of the subsidiary and there was no other nonvoting stock out-standing, the control requirement of Section 351 would not be met, but the subsidiary could still consolidate with its parent. *See* Manning, *supra* note 14, at 191.

[26] DEL. CODE ANN. tit. 8, § 160(c) (1991).

Delaware extends this restriction to shares held by a subsidiary if a majority of the subsidiary's voting shares are held, directly or indirectly, by the issuer;[27] similar provisions appear in most state corporate statutes[28] as well as the Model Business Corporation Act.[29]

These voting restrictions on affiliate-held stock apply in an all-or-nothing way. If the issuer holds a majority of the affiliate's voting stock, then none of the stock of the issuer held by the affiliate may be voted; if the issuer holds less than a majority, all of this stock may be voted. Thus, there is no proportionate attribution of voting power. The Delaware statute, unlike the Model Business Corporation Act, applies regardless of whether the majority interest in the affiliate is held "directly or indirectly"; presumably, indirect ownership counts if it gives the issuer control over the affiliate.[30]

In Figure 1 the two corporations own a majority of the shares of each other's voting stock. Consequently, these disenfranchisement rules would deprive each of these corporations the right to vote each other's stock, and the only shares that could be voted would be the shares owned by I_1 and I_2. While this outcome might seem reasona-

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[27] *Id.*

[28] See, e.g., CAL. CORP. CODE § 703(b); NEB. REV. STATS. § 21-2033; N.Y. BUS. CORP. LAW § 612(b); N.J. REV. STATS. § 14A:5-13; N.M. STATS. ANN. § 53-11-33(B); TEX. BUS. CORP. ANN. § 2.29(B); WASH. REV. CODE § 23B.07.210(2). Some states relax the rule prohibiting a subsidiary from voting shares of its parent in "special circumstances." IOWA CODE § 490.721(2); MISS. CODE ANN. § 79-4-7.21(b); UTAH BUS. CORP. ACT § 16-10a-721(2). The only jurisdictions with corporate statutes that do not restrict a subsidiary from voting the shares of its parent are the District of Columbia, Illinois, Massachusetts, Missouri, Ohio, Oklahoma, and Pennsylvania. REV. MOD. BUS. CORP. ACT. ANN. 7-99 (3rd ed. 1994).

[29] MODEL BUSINESS CORP. ACT § 33(2), § 33(2) ¶ 3 (note on statutory provisions) (Am. Ba. Ass'n Comm. on Corp., Banking and Bus. Law 1971 & Supp. 1979).

[30] See Speiser v. Baker, 525 A.2d 1001, 1007–08 (Del. Ch. 1987); Kalmanovitz v. G. Heileman Brewing Co., 595 F. Supp. 1385, 1398–99 (D. Del. 1984), *aff'd*, 769 F.2d 152 (3d Cir. 1985).

ble,[31] it is annihilating where a wholly-owned subsidiary acquires a majority of its parent shares, as shown in Figure 2.

FIGURE 2
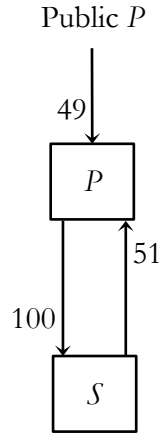
In such a case, none of the subsidiary's shares would be entitled to vote. Who would elect its directors?

Another anomaly arises, at least under Delaware law, whenever two corporations own a majority in interest in each other (as in Figure 1). Delaware's disenfranchisement provision reads as follows:

> Shares of its own capital stock belonging to the corporation or to another corporation, if a majority of the shares entitled to vote in the election of directors of such other corporation is held, directly

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[31] One context where this outcome does not seem reasonable is if a structure like that shown in Figure 1 arises as a result of a "Pac Man" tender offer defense, in which the target launches a tender offer for shares of the acquiror in response to the acquiror's previous offer for the target's shares. If there is a sufficient response to both offers, each party could end up owning a majority of the other's shares, causing them to become nonvoting. In that event, the only voting shares are those held by the nontendering shareholders, who may be indifferent, apathetic or ill-informed. See Deborah A. De Mott, *Pac-Man Tender Offers*, 1983 Duke L.J. 116.

or indirectly, by the corporation, shall neither be entitled to vote nor be counted for quorum purposes.[32]

This provision applies only if the issuer holds a majority of the affiliate's shares that are entitled to vote. Yet, if the affiliate also holds a majority of the issuer's shares, then the issuer is not entitled to vote the shares of the affiliate. By its literal terms, the statute undermines its application in both directions. It applies if and only if it does not apply.[33] Such a result might well be regarded as a strange loop in the law.[34]

These disenfranchisement statutes have potential tax consequences, since the tax treatment of stock can depend on its voting rights. In Revenue Ruling 73-28,[35] corporation X acquired all of the stock of its second-tier subsidiary Z from its first-tier subsidiary Y in exchange for its own voting stock. The Service ruled that the transaction was a valid B reorganization, regardless of whether under state law Y could vote the shares of X stock received in the exchange.[36]

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[32]   DEL. CODE ANN. tit. 9, § 160(c) (1991).

[33]   De Mott, *supra* note 31, at 119–20. California's statute arguably avoids this paradox by referring to the ownership of a majority of voting shares, rather than entitlement to vote them, but that interpretation is not entirely clear. CAL. CORP. CODE § 703(b) (West 1990).

[34]   For Douglas Hofstadter's own examples of strange loops in the law, *see* Hofstadter, *supra* note 3, at 692–93.

[35]   1973-1 C. B. 187.

[36]   See also Manning, *supra* note 14, at 200.

III. A Formal Theory of Strange Loops

So far, the strange loops presented here involved only two corporations. There is no real limit, however, to the number of corporate links in a strange loop, or to the number of strange loops that can form cross linkages in a corporate lattice. To analyze these systematically, some formalization of these relationships is needed. This Part presents a notation for recording strange loops, and a formal theory for analyzing the relationships they create.

The formal theory expresses abstractly relationships among corporations and their shareholders. The term "theory" in this sense has much in common with the term in its scientific sense, in that it presents a way of looking at some aspect of the world that helps us to understand it better. Conclusions within the formal theory can be expressed and proven with certainty as mathematical theorems. This does not mean, however, that the formal theory is necessarily a valid way of looking at corporate-shareholder relationships. How well the formal theory applies to the real world of corporations and shareholders is an empirical question, more suited to the expertise of lawyers than mathematicians.

A. *The Toolbox*

There is a finite set $\{C\}$ of CORPORATIONS,[37] and $\{I\}$ of INDIVIDUALS. A function $A(C_i)$, called ASSET VALUE, which is equal to or greater than zero for all i, represents the fair market value of a corporation's underlying assets (excluding shares of other corporations in

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[37] Terms introduced in block capitals are meant to evoke concepts outside the formal theory, but are not part of it. Perfectly consistent interpretations of the theory exist in which the elements of $\{C\}$ bear no resemblance to corporations; one such interpretation is offered in Part III.D *infra* (p. 191). A complete listing of these terms appears in Appendix II.

{C}). Another function $S(C_i, C_j)$ with two arguments, called SHARE-HOLDING, which takes on values between zero and one, represents the direct percentage stock ownership of one corporation in another. A similar function $M(I_k, C_j)$ is the same as S, except that the first argument is a member of {I}: Individuals can be holders, but not issuers, of shares. Treasury stock is disregarded by treating $S(C_i, C_i)$ as equal to zero.

A function $V(C_i)$, called VALUE, is defined as follows:[38]

(1) $$V(C_i) = A(C_i) + \sum_{j \neq i} S(C_i, C_j) V(C_i).$$

According to (1), the value of each corporation is its asset value plus the sum of the value of each other corporation times its level of shareholding in that other corporation. If {C} has n members, then (1) gives rise to n simultaneous equations in n unknowns.

There is a constraint on the values of M and S:

Condition 1:   For all j, $\sum_{k} M(I_k, C_j) + \sum_{i \neq j} S(C_i, C_j) = 1$.

This condition requires that the outstanding shareholding of each corporation be 100%.

One final constraint is needed to ensure the existence and uniqueness of a solution:

Condition 2: Each subset of {C} contains at least one member C_j for which either $S(C_i, C_j) > 0$ for some C_i outside the subset, or $M(I_k, C_j) > 0$ for some I_k.

Condition 2 requires that no group of corporations be wholly owned by each other.

A corollary of Condition 2 is that at least one corporation must have an individual shareholder. This can be seen by taking the subset to be the entire set {C}, in which case there must be a C_j and I_k for

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[38] Except as otherwise indicated, all summation indices for corporations run from one to the number of corporations in {C}, and all summation indices for individuals run from one to the number of individuals in {I}.

which $M(I_k,C_j) > 0$. It should be stressed that an "individual," as used here, does not have to be a shareholder that is a person. An "individual" can also be an entity that is not owned, directly or indirectly, by any other persons or entities under consideration. The ownership of such an entity is irrelevant to any strange loops among the corporations in $\{C\}$, and therefore such entities are equivalent to individuals for purposes of the formal theory.

B.   *Solving the Equations*

The example at the beginning of the article now can be solved by expressing the relationships in the diagram in terms of the equations of the formal theory. Since C_1's asset is worth $100 and C_2's asset is worth $200,

(2) $A(C_1) = 100, A(C_2) = 200.$

The 90% cross ownership is expressed as follows:

(3) $S(C_1,C_2) = 0.9, S(C_2,C_1) = 0.9.$

These values can be substituted into Equation (1):

(4) $V(C_1) = 100 + 0.9 \times V(C_2), V(C_2) = 200 + 0.9 \times V(C_1).$

Substituting the right-hand side of the first equation in (4) for $V(C_1)$ in the second gives:

(5) $V(C_2) = 200 + 0.9 \times (100 + 0.9 \times V(C_2)).$

Equation (5), when solved for $V(C_2)$, yields a value of $1,526 for $V(C_2)$. This value of $V(C_2)$ then can be substituted in the first equation in (4):

(6) $V(C_1) = 100 + 0.9 \times 1,526 = 1,474.$

One result that initially might seem curious is that the sum of $V(C_1)$ and $V(C_2)$ is $3,000, which is 10 times the value of the underlying assets. Of course, no real value is being created: The values of the corporations reflect both direct and indirect ownership of the same

assets. The apparent increase in value is comparable to the overlap in value that occurs when the constituent corporations of an ordinary corporate tree without strange loops are valued separately. Since only one-tenth of the outstanding stock of C_1 and C_2 is held by individuals, the total value of their stock is $300, which is equal to the underlying asset value.

Although the extensive cross ownership effectively scrambles the interests of the two individuals in the underlying assets, the values of the shares held by each are not quite equal. I_2 has the slight advantage: His shares are worth 10% of $1,526, or $153, while I_1's shares are worth 10% of $1,474, or $147. Strictly speaking, the formal theory is measuring the value of each individual's percentage interest of the underlying assets. The shares that evidence this interest might be worth more (for example, if they were more marketable than the underlying assets) or they might be worth less (for example, because of a minority interest discount, or because of corporate-level taxes).

C.   *The General Solution*

Although the ad hoc algebra in the preceding section solved the equations in (1) fairly quickly for a two-corporation case, an approach that will work for any number of corporations must be more systematic. This section briefly reviews, in the context of the formal theory, a standard procedure for solving sets of simultaneous equations, making it possible to compute values of ownership interests in situations involving any number of corporations and strange loops.[39] Readers familiar with this procedure can skim this section quickly.

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[39]   The procedure is based on the method of row-reduced echelon matrices. *See* KENNETH HOFFMAN & RAY KUNZE, LINEAR ALGEBRA 1–16 (2d ed. 1971). The approach is described with a more practical emphasis in GEORGE B. THOMAS, JR., CALCULUS AND ANALYTIC GEOMETRY 435–41 (4th ed. 1969), and with a more theoretical emphasis in SAUNDERS MACLANE & GARRETT BIRKHOFF, ALGEBRA 212–19 (3d ed. 1988).

First, put the constant $A(C_i)$ on the right-hand side of the equal sign in (1), and everything else on the left:

$$(7) \qquad V(C_i) - \sum_{j \neq i} S(C_i, C_j) V(C_j) = A(C_i).$$

Then write the equations in a series of rows, with the summations opened up, and each term involving a particular $V(C_i)$ directly above or below each term involving the same $V(C_i)$ in the other rows:

(8)

$$
\begin{array}{ccccccc}
V(C_1) + & -S(C_1, C_2) V(C_2) + \cdots + & -S(C_i, C_n) V(C_n) = & A(C_1) \\
-S(C_2, C_1) V(C_1) + & V(C_2) + \cdots + & -S(C_2, C_n) V(C_n) = & A(C_2) \\
\vdots & \vdots \quad \ddots & \vdots & \vdots \\
-S(C_n, C_1) V(C_1) + & -S(C_n, C_2) V(C_2) + \cdots + & V(C_n) & A(C_n)
\end{array}
$$

Progress towards a solution takes the form of successively replacing individual equations with other equations that are equivalent in the sense that the solution of the altered set is the same as the solution of the original set. The rules for permitted replacements flow from the high-school maxim, "what you do to one side of the equation, do also to the other." For example, an equation can be replaced by a scaled version of itself, determined by multiplying both sides by the same factor. Also, an equation can be replaced by the sum of itself and a scaled version of another equation. This last operation is made easier by the orderly arrangement in Equation (8), in which all the terms involving a particular $V(C_i)$ appear in the same column.

The mechanics can be simplified, without loss of information, by replacing the table of equations with a matrix showing only the coefficients. The jth column of the ith row of this matrix is the coefficient for $V(C_j)$ in the ith equation; the last column (after the vertical bar) has $A(C_i)$ in the ith row.

$$\left[\begin{array}{cccc|c} 1 & -S(C_1,C_2) & \cdots & -S(C_1,C_n) & A(C_1) \\ -S(C_2,C_1) & 1 & \cdots & -S(C_2,C_n) & A(C_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -S(C_n,C_1) & -S(C_n,C_2) & \cdots & 1 & A(C_n) \end{array}\right]$$

The secret is to transform the part of this matrix to the left of the vertical bar into one in which all of the coefficients $-S(C_i,C_j)$ become zero, leaving a diagonal string of ones. In the process, the coefficients to the right of the vertical bar will be transformed into the correct values of $V(C_i)$.

The matrix is transformed one column at a time, working from left to right. The first column is turned into a column of zeros (below the topmost one) by replacing the second and each subsequent row by the sum of it and the first row scaled by the first element of the row being replaced. Then the second column is addressed, first by scaling the second row so that its second element is again one, and then replacing each row (other than the second) by the sum of it and the second row scaled by the second element of the row being replaced. And so forth.

The process is easiest to see by returning to the example in the last section. The matrix starts out as follows:

$$\left[\begin{array}{cc|c} 1 & -0.9 & 100 \\ -0.9 & 1 & 200 \end{array}\right]$$

The leftmost element of the second row is turned into zero by replacing the second row with the sum of it and the first row scaled by 0.9:

$$\left[\begin{array}{cc|c} 1 & -0.9 & 100 \\ 0 & 0.19 & 290 \end{array}\right]$$

The second row is rescaled so its second element is one:

$$\left[\begin{array}{cc|c} 1 & -0.9 & 100 \\ 0 & 1 & 1{,}526 \end{array}\right]$$

Finally, the second element of the first row is turned into zero by replacing the first row with the sum of it and the second row scaled by 0.9:

$$\begin{bmatrix} 1 & 0 & | & 1{,}474 \\ 0 & 1 & | & 1{,}526 \end{bmatrix}$$

The answers are then read off the last column.

Another example illustrates the application of this method to three corporations. Suppose C_1, C_2 and C_3 own assets (excluding stock of each other) worth $75, $50, and $100 respectively. In addition, C_1 owns 10% of the stock of C_2, which in turns owns 20% of the stock of C_3, which in turn owns 30% of the stock of C_1. The remaining stock of the three corporations is held by three individuals, as shown in Figure 3.

FIGURE 3
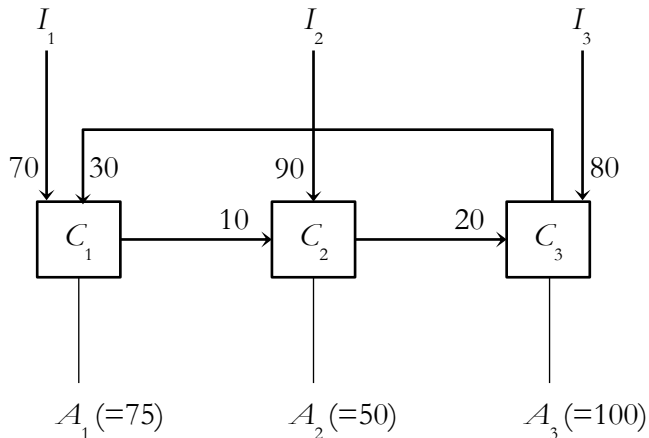
The basic equations in (*1*) are filled in as follows:

(9)
$$V(C_1) = 0.1 \times V(C_2) + 75$$
$$V(C_2) = 0.2 \times V(C_3) + 50$$
$$V(C_3) = 0.3 \times V(C_1) + 100$$

FIGURE 4

$$\left[\begin{array}{ccc|c} 1 & -0.1 & 0 & 75 \\ 0 & 1 & -0.2 & 50 \\ -0.3 & 0 & 1 & 100 \end{array}\right]$$

Starting matrix

(A)

$$\left[\begin{array}{ccc|c} 1 & -0.1 & 0 & 75 \\ 0 & 1 & -0.2 & 50 \\ 0 & -0.03 & 1 & 122.5 \end{array}\right]$$

Replace row 3 with the sum of itself
and 0.3 times row 1

(B)

$$\left[\begin{array}{ccc|c} 1 & 0 & -0.02 & 80 \\ 0 & 1 & -0.2 & 50 \\ 0 & -0.03 & 1 & 122.5 \end{array}\right]$$

Replace row 1 with the sum of itself
and 0.1 times row 2

(C)

$$\left[\begin{array}{ccc|c} 1 & 0 & -0.02 & 80 \\ 0 & 1 & -0.2 & 50 \\ 0 & 0 & 0.994 & 124 \end{array}\right]$$

Replace row 3 with the sum of itself
and 0.03 times row 2

(D)

$$\left[\begin{array}{ccc|c} 1 & 0 & -0.02 & 80 \\ 0 & 1 & -0.2 & 50 \\ 0 & 0 & 1 & 124.75 \end{array}\right]$$

Scale row 3 up by a factor
of 0.994

(E)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 82.49 \\ 0 & 1 & -0.2 & 50 \\ 0 & 0 & 1 & 124.75 \end{array}\right]$$

Replace row 1 with the sum of itself
and 0.02 times row 3

(F)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 82.49 \\ 0 & 1 & 0 & 74.95 \\ 0 & 0 & 1 & 124.75 \end{array}\right]$$

Replace row 2 with the sum of itself
and 0.2 times row 3

(G)

$82.49 = 0.1 \times 74.95 + 75$

$74.95 = 0.2 \times 124.75 + 50$

$124.75 = 0.3 \times 82.49 + 100$

The solution fits the equations

(H)

Figure 4 shows the sequence of matrices that solve the equations. The computations are somewhat laborious, and become rapidly more so as the number of corporations increases. Fortunately, the process is completely mechanical, and can be done easily by a personal comput-

er. An alternative format, which can be handled by a spreadsheet program, is offered in Part IV below.

D. *Two Theorems About Strange Loops*

This Part asserts two propositions: First, that the value of shares held by individuals is equal to underlying asset value; and second, that the simultaneous equations that determine share values always have a unique solution. While both propositions may seem intuitively plausible, intuition can be unreliable where strange loops are concerned. Hence, both propositions are presented here as theorems of the formal theory. Proofs are offered in Appendix I.

The first theorem establishes that no value is created or destroyed by the system: No matter how byzantine the cross ownership, the total value of the shares held by individuals is equal to the total underlying asset value of the corporations. The value of individual I_k's shareholding in corporation C_j is $M(I_k,C_j)V(C_j)$. Since the asset value of corporation C_j is $A(C_j)$, this theorem can be stated as follows:

Theorem 1: The total value of the shares of corporations in $\{C\}$ held by individuals is equal to the asset value of the corporations:

$$\sum_j \left(\sum_k M(I_k,C_j)V(C_j) \right) = \sum_j A(C_j).$$

Not all sets of simultaneous equations have unique solutions. In some cases, the equations are mutually inconsistent, with no common solution; in others, the equations are redundant and therefore insufficient to narrow the number of possibilities down to one. The second theorem asserts that the procedure outlined above for solving simultaneous equations generates a unique solution in precisely those instances where Condition 2 holds, that is, where no group of corporations is collectively self-owned.

Theorem 2: The equations in (8) have a unique solution if and only if Condition 2 is true.

Because the development of the formal theory was motivated by corporate strange loops, it is tempting to think that the theory has a "correct" interpretation, in which each C_i is a corporation, each I_k is an individual, and $M(I_k, C_i)$ is a percentage of direct share ownership. Yet, the validity of the theorems presented above is independent of any issues concerning the application of the theory to real corporations and individuals. Indeed, the theorems apply equally to more fanciful interpretations.

For example, imagine a neural network. Each C_i represents a neuron; each I_k a detector. $M(I_k, C_i)$ represents the synaptic strength between neuron i and detector k, while $S(C_i, C_j)$ represents the synaptic strength between two neurons. $A(C_i)$ is the stimulus applied to C_i, which is transmitted to other neurons in proportion to the synaptic strength of their relations with C_i. Condition 1 ensures that the amplitude of the overall stimulus remains constant, and Condition 2 ensures that each stimulus eventually reaches one or more detectors.

Thus, even though the theorems are presented in terms of "individuals" and "corporations," these labels were attached to the theoretical objects because that is how the theory is being applied here, not because individuals and corporations are somehow intrinsic to the theory. Indeed, all that is intrinsic to the theory are the mathematical relationships, and while the truth of the theorems is a mathematical certainty, this certainty is a feature only of the mathematical content, and not of any particular application. Whether the theorems are true of corporations (or of neurons, for that matter) is an empirical question about where it is appropriate to apply the theory.

The original interpretation, however natural, is not without its problems, which sort out into two kinds. First, for simplicity, the formal theory omits elements that correspond to real world items like liabilities and preferred stock. Second, the formal theory embodies assumptions that may be inaccurate even in the simple cases that the theory explicitly covers. For example, Equation (1) embodies the as-

sumption that the value of a corporation's stock is equal to its net asset value. This particular assumption can be false for many reasons, including the tax system itself. In some of the applications described below, therefore, it is worth considering whether the implications of the formal theory remain valid when these assumptions are relaxed.

IV. Loss Carryforward Limitations Under Section 382

A. *Existence of an Ownership Change*

Section 382 limits the use of a corporation's net operating loss carryforwards following an "ownership change," which occurs if the ownership of more than 50% of its stock changes during a three-year testing period.[40] Percentage ownership is determined on the basis of value,[41] and a set of attribution rules tracks changes, to the greatest extent possible, at the highest level of beneficial ownership.[42] These rules generally attribute stock held by an entity to an entity's owners, in proportion to their interests in the entity. A *de minimis* rule provides that if, in the course of attributing stock to an entity, the entity is deemed to own less than 5% of the stock of the loss corporation, then the attribution stops there, and there is no further attribution of its stock ownership up the chain to its owners.[43]

The application of these rules to a typical corporate tree is usually straightforward: Only changes in ownership of the highest-tier entities are taken into account in determining whether a lower-tier entity has had an ownership change. Strange loops, however, are another matter. Consider the example in Figure 1, where two corporations own 90% of the stock of each other. Absent attribution, a sale by I_1 of her 10% interest in C_1 would, by itself, fail to trigger an ownership change. Suppose, however, both I_1 and I_2 sold their stock to unrelated parties. Since I_2 owns 10% of C_2 and C_2 owns 90% of C_1, I_2 indirectly owns 9% of C_1. So far, a sale by both I_1 and I_2 amounts to a change in own-

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40 I.R.C. § 382(a), (b), (g).

41 I.R.C. § 382(k)(6)(C).

42 I.R.C. § 382(*l*)(3)(A); Temp. Treas. Reg. § 1.382-2T(h).

43 Temp. Treas. Reg. § 1.382-2T(h)(2)(iii)(A). This *de minimis* rule does not apply if ownership interests were organized deliberately to take advantage of it. Temp. Treas. Reg. § 1.382-2T(k)(4).

ership of 19% of the stock of C_1, which is still insufficient for an ownership change. This is plainly the wrong result. If the only two people with a beneficial interest in an entity sell all of their stock, that entity has to have suffered an ownership change.

1.   *Percentage Ownership: The Two-Corporation Case*

What is needed is a concept of percentage ownership that reflects Section 382's stock attribution rules and takes strange loops properly into account. One approach is to compute the percentage interest of an individual in a corporation by adding the direct and indirect interests via every possible chain of stock ownership, including chains that pass through strange loops.

In measuring I_1's percentage interest in C_1 in Figure 1, the shortest chain has a single link, representing I_1's direct 10% shareholding in C_1. The next chain goes from C_1 to C_2 and back to C_1, taking one circuit around the strange loop. I_1 indirectly owns 8.1% of C_1 via this chain of attribution, since I_1 owns 10% of C_1, C_1 owns 90% of C_2, and C_2 owns 90% of C_1. Moreover, an additional 6.56% can be attributed to I_1 under the next chain of attribution, which takes another circuit around the strange loop.

The total of the percentages computed via each chain of stock attribution is finite, even though there are an infinite number of such chains. The attributed interests for the chains form a geometric series, the PERCENTAGE INTEREST:

(10) $P(I_1,C_1) = 0.1 + (0.1 \times 0.9^2) + (0.1 \times 0.9^4) + \ldots$

Each term is smaller than the preceding one by a factor of 0.9^2, or 81%.

The formula for the sum of a geometric series starting with a and decreasing by a constant factor r is $a/(1 - r)$.[44] Tax lawyers use this

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[44]   This formula is as easy to derive as it is to remember. The sum S is:

formula instinctively when drafting provisions for the borrower's payment of interest to a foreign lender free of withholding tax, where the payments must cover, in addition to the base amount of interest, the withholding tax itself, plus the tax on the tax, plus the tax on the tax on the tax, and so forth. The necessary payment is simply the base amount grossed up by dividing it by 100% minus the tax rate. Indemnity payments that hold the indemnitee harmless on an after-tax basis are computed the same way.

This formula can be used to compute $P(I_1,C_1)$:

(11) $P(I_1,C_1) = 0.1/(1 - 0.81) = 0.526.$

I_1's percentage interest in C_2 can be computed in the same fashion. The chains of attribution are the same as those used in computing $P(I_1,C_1)$, except each chain has an additional link from C_1 to C_2. Since $S(C_1,C_2) = 0.9$, it follows that $P(I_1,C_2) = 0.9 \times P(I_1,C_1)$, which is 0.474. The percentage interest of I_2 in C_1 and C_2 can be deduced immediately from the symmetry of the example, and the results are summarized in Figure 5.[45] Note that the combined percentage of the interests of I_1 and I_2 in the two corporations, when computed in this manner, is 100%.

In simple cases, there is a way to determine percentage ownership without cycling endlessly through strange loops. One can treat the degree of self-ownership represented by the strange loop as a kind of treasury stock. The indirectly self-owned interest is then viewed, like treasury stock, as not outstanding. In our two corporation example, C_1

(i) $S = a + ar + ar^2 + ar^3 + ar^4 + \ldots$
Multiply both sides by r:
(ii) $rS = ar + ar^2 + ar^3 + ar^4 + \ldots$
When Equation (ii) is subtracted from Equation (i), all of the terms on the right cancel, except the first:
(iii) $S - rS = a.$
Solving Equation (iii) for S gives the formula:
(iv) $S = a/(1 - r).$

[45]   This approach for a two-corporation strange loop also is used by Canellos, *supra* note 21, at 4.

owns 90% of C_2, which owns 90% of C_1, so C_1's indirect ownership interest in itself is 90% of 90%, or 81%. Of the 19% of C_1's stock that is not indirectly self-owned, I_1 owns 10% directly, and I_2 owns 9% through its interest in C_2. So I_1's interest is 10/19ths, or 52.6%, and I_2's interest is 9/19ths, or 47.4%. These percentages agree with the percentages calculated with infinite looping.

FIGURE 5

Percentage Interest	C_1	C_2
I_1	52.6	47.4
I_2	47.4	52.6
Total	100.0	100.0

The table in Figure 5 answers the first two questions previously posed with regard to Section 382. A sale by I_1 of her interest causes an ownership change of C_1 but not of C_2, since only I_1's interest in C_1 exceeds the 50% threshold required for an ownership change to occur.[46] This assumes, of course, that the percentage interests, as computed here, are consistent with how percentage interests should be measured for purposes of Section 382.

The Section 382 stock attribution rules refer to Section 318, which provides that stock owned by a corporation is attributed to its shareholders in proportion to the value of their shareholdings.[47] For corporate trees without strange loops, the percentage interest function accurately reflects shareholder attribution for purposes of Section

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[46] C_2 also might have an ownership change if the two corporations join in filing consolidated returns, with C_1 as the common parent. *See infra* note 58 and accompanying text. Whether they should be entitled to consolidate is discussed in Part V (p. 217).

[47] I.R.C. § 318(a)(2)(C). Section 318 itself provides for attribution only to shareholders with a 50% or greater interest, but this 50% limitation does not apply for purposes of Section 382. I.R.C. § 382(*l*)(3)(A)(ii)(I).

382, except for the *de minimis* rule that blocks attribution from entities with less than a 5% interest in the loss corporation.[48]

The regulations under Section 318 address, in at least one circumstance, the possibility that stock ownership might be attributed to the issuer, but this circumstance does not involve a strange loop. Section 318 provides for "back" attribution to a corporation of shares held by a shareholder with a 50% or greater interest.[49] The regulations provide that this form of attribution shall not cause the ownership of the issuer's own shares held by the shareholder to be attributed back to the issuer.[50] This restriction has no direct bearing on Section 382, however, because back attribution applies for Section 382 purposes only to the extent provided by regulations, and no regulations have been issued yet that provide for back attribution.[51]

2. *Percentage Ownership: The General Case*

The preceding Part presented two consistent ways of calculating percentage ownership, with stock attribution, for two corporations with cross-ownership interests. The first approach cycled endlessly through the strange loop, while the second "pinched off" the strange loop by treating the indirect self-owned interests as not outstanding. When there are multiple interconnected strange loops, there is no clear way to chart lines of attribution under the first method, or to compute the indirect self-owned interest under the second method.

The difficulties can be seen by modifying the 90% cross owner-ship in the original example so that only 80% is direct, and the other

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[48]   Temp. Treas. Reg. § 1.382-2T(h)(2)(iii)(A).

[49]   I.R.C. § 318(a)(3)(C).

[50]   Treas. Reg. § 1.318-l(b)(l). This type of attribution would not be blocked by the "anti-sidewise attribution" rule of I.R.C. § 318(a)(5)(C), because that rule re-stricts only back attribution of shares attributed to a shareholder by reason of its interest in another entity.

[51]   I.R.C. § 382(*l*)(3)(A)(iii); Temp. Treas. Reg. § 1.382-2T(h)(3).

10% is held through a third corporation, C_3, which is owned equally by C_1 and C_2, as shown in Figure 6.

FIGURE 6
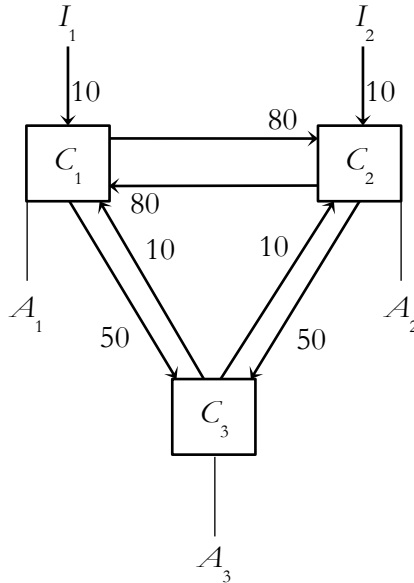
It is hard to say intuitively whether this modification increases I_1's percentage ownership of C_1, decreases it or leaves it the same. At first glance, it appears that C_1 owns 64% of itself indirectly, through the strange loop through C_2 (80% of 80%), and another 5% through the strange loop through C_3 (50% of 10%). Yet, there is a third strange loop running through both C_2 and C_3, suggesting another 4% self-owned interest (80% of 50% of 10%). These calculations, however, track ownership interests through C_2 and C_3, without regard to the strange loop between them that does not include C_1.

The percentage interest of I_1 and I_2 in each of these three corporations can be determined by calculating the percentages of income of each corporation that is ultimately distributable to each shareholder, assuming that the income flows through the corporations and to the shareholders without any amount lost to taxes. (While this assumption is contrary to fact in many cases, the flow of income acts

here as a proxy for attribution of stock ownership, and therefore, these taxes can be properly disregarded.) To perform the calculation, the key is to have the proper notation with which to organize the relevant data. The matrix notation introduced in Part III as a convenient way to list a series of equations is now indispensable. Let D represent the ordered set $<d_1, d_2, \ldots, d_n>$, where d_i is the amount that corporation C_i has to distribute. Such an ordered set is a matrix with a single row (or it can be turned sideways and shown as a matrix with a single column), referred to as a vector.[52]

If C_j pays a dividend, the portion of the dividend received by another corporation C_i is $S(C_i, C_j)$, the percentage stock ownership of corporation C_i in corporation C_j. Let S be a matrix with $S(C_i, C_j)$ as the element in its ith row and jth column. For the corporate chart in Figure 6, the matrix S is:

$$S = \begin{bmatrix} 0 & 0.8 & 0.5 \\ 0.8 & 0 & 0.5 \\ 0.1 & 0.1 & 0 \end{bmatrix}.$$

A matrix is more than a table of numbers. Matrices are mathematical objects in their own right, and can be added, multiplied and in some cases, raised to a power. Matrix addition is straightforward: The sum of two matrices of the same size is a matrix with each element equal to the sum of the corresponding elements of the matrices being added. (Matrices of different sizes cannot be added.) Matrix multiplication seems convoluted at first: When two matrices are multiplied, the element in the ith row and jth column of the product matrix is a so-called "dot product" of the ith row of the first matrix times the jth column of the second matrix. A dot product of two vectors (a row or

[52]   This definition converges with the familiar notion of a vector as a directed line segment, when the ordered set is interpreted as Cartesian coordinates in n-dimensional space, and the line segment is drawn from the origin to the point represented by those coordinates. The vector then can be taken to be any segment with the same length and orientation, or more precisely, the equivalence class of all such segments.

column of a matrix is a vector) is determined by multiplying their components element-by-element and adding up the products. Since a dot product is defined only for vectors with the same number of elements, two matrices can be multiplied only if the number of columns in the first matrix is the same as the number of rows in the second. In particular, two square matrices of the same size can be multiplied together, and a square matrix can be raised to a power by multiplying it by itself a number of times. A product matrix has the same number of rows as the first matrix and the same number of columns as the second.

The matrices S and D described above can be multiplied together, since S is a square matrix with a number of columns equal the number of corporations under consideration, and D can be expressed as a column vector with an equal number of "rows" (that is, elements). Suppose C_1 has \$750 to distribute, C_2 has \$500 and C_3 has \$1,000. D can be expressed as follows:

$$D = \begin{bmatrix} 750 \\ 500 \\ 1,000 \end{bmatrix}$$

When S and D are multiplied together, the product is

$$SD = \begin{bmatrix} 0 & 0.8 & 0.5 \\ 0.8 & 0 & 0.5 \\ 0.1 & 0.1 & 0 \end{bmatrix} \times \begin{bmatrix} 750 \\ 500 \\ 1,000 \end{bmatrix} =$$

$$\begin{bmatrix} 0 \times 750 + 0.8 \times 500 + 0.5 \times 1,000 \\ 0.8 \times 750 + 0 \times 500 + 0.5 \times 1,000 \\ 0.1 \times 750 + 0.1 \times 500 + 0 \times 1,000 \end{bmatrix} = \begin{bmatrix} 900 \\ 1,100 \\ 125 \end{bmatrix}$$

This product SD indicates how much each corporation receives when the distributions represented by D are paid. For example, C_1 receives nothing from itself, but receives 80% of the \$500 paid by C_2, or \$400, plus 50% of the \$1,000 paid by C_3, or \$500, for a total of \$900.

To solve the rest of the problem, one must imagine infinitely repeated synchronized dividends. After each corporation makes a distribution, it then makes a second distribution equal to what it received in the first distribution. A third distribution is made out of the proceeds of the second distribution. And so on, forever, but with ever dwindling amounts, because funds leak out of the system to the individual shareholders as each distribution is made. (Recall that Condition 2 requires that at least one corporation in every set have a shareholder outside the set.)

Define D_1 to be the amount received on the first distribution, which becomes the amount paid on the second. As we have seen, $D_1 = SD$. When D_1 is paid, the amount received by each corporation is SD_1 which is equal to $S(SD)$. Call this amount D_2. When D_2 is paid, each corporation receives SD_2, or $S(S(SD))$ and so forth.

Matrix multiplication follows some, but not all, of the laws of regular multiplication. For example, matrix multiplication is associative, but not commutative. Thus, it is always true that $A(BC) = (AB)C$, but it is not always, or even typically, true that $AB = BA$. Consequently, when doing matrix algebra, it is important to keep the order of the terms straight. Matrix multiplication follows the distributive law; indeed, because there is no commutativity, a left distributive law $A(B + C) = AB + AC$ can be distinguished from a right distributive law $(A + B)C = AC + BC$.

The associative law can be used to describe the ith distribution D_i. For example,

(12)
$$D_2 = SD_1 = S(SD) = (SS)D = S^2D,$$
$$D_3 = SD_2 = S(S^2D) = (SS^2)D = S^3D \ldots$$

In general,

(13)
$$D_i = S^iD.$$

A matrix M can be constructed that shows how much each individual shareholder receives upon each distribution with $M(I_i C_j)$ as the element in its ith row and jth column. For the example in Figure 6,

$$(14) \qquad M = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \end{bmatrix}$$

Just as the product SD shows the amount that each corporation receives from a distribution, the product MD is a vector R_0 that shows how much each individual receives. In the example,

$$(15) \quad MD = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \end{bmatrix} \times \begin{bmatrix} 750 \\ 500 \\ 1,000 \end{bmatrix} =$$

$$\begin{bmatrix} 0.1 & \times & 750 & + & 0 & \times & 500 & + & 0 & \times & 1,000 \\ 0 & \times & 750 & + & 0.1 & \times & 500 & + & 0 & \times & 1,000 \end{bmatrix} = \begin{bmatrix} 75 \\ 50 \end{bmatrix}$$

Thus, out of this set of distributions, I_1 receives \$75 and I_2 receives \$50. As successive distributions D_1 are made, the vector R_i that shows the amount each individual receives is given by MD_i, which, by Equation (13), is equal to MS^iD.

This sequence of distributions is assumed to be infinitely repeated. The total amount that each individual receives can be represented by the sum R of the vectors R_i that report how much each individual receives from each distribution:

$$(16) \qquad R = \sum_{i=0}^{\infty} R_i = \sum_{i=0}^{\infty} MS^iD.$$

To interpret this summation, it is necessary to understand the expression S^0, which occurs in the first term. Just as any number raised to the zero-th power is equal to one, any square matrix raised to the zero-th power is equal to the identity matrix, which is a square matrix of the same size that has ones on its main diagonal and zeroes everywhere else. This matrix, represented by I, has the property that $AI = A$ for every matrix A that can be right-multiplied by I, and $IB = B$ for every matrix B that can be left-multiplied by I. Thus, $S^0D = ID = D$, the amount of the initial distribution.

In the summation shown on the right of Equation (16), M appears in each term, and can be placed to the left of the summation sign using the left distributive law. D also appears in each term, but it cannot be placed to the left of the summation sign, because matrix multiplication is not commutative. D can, however, be placed to the right of the summation as a whole, using the right distributive law:

$$(17) \qquad R = M\left(\sum_{i=0}^{\infty} S^i D\right) = M\left(\sum_{i=0}^{\infty} S^i\right) D.$$

All that remains is to compute the term in the parentheses, which is an infinite sum of square matrices S^i. This series of matrices is analogous to a geometric series of ordinary numbers, and can be solved by a procedure analogous to that used in note 44 to arrive at the formula for the sum of such a series. Care must be taken, however, to apply the laws of arithmetic in a manner consistent with the particular requirements of matrix algebra.

Call the sum of the series matrix F:

$$(18) \qquad F = \sum_{i=0}^{\infty} S^i.$$

Multiply both sides of Equation (18) on the left by S, and use the left distributive law to put the additional factor S inside the summation:

$$(19) \qquad SF = S\left(\sum_{i=0}^{\infty} S^i\right) = \sum_{i=0}^{\infty} S^{i+1} = \sum_{i=1}^{\infty} S^i.$$

Now subtract the leftmost side of Equation (19) from the left side of Equation (18), and the rightmost side of Equation (19) from the right side of Equation (18):

$$(20) \qquad F - SF = \sum_{i=0}^{\infty} S^i - \sum_{i=1}^{\infty} S^i.$$

On the right side of Equation (20) all of the terms cancel out except S^0, which is the identity matrix I:

$$(21) \qquad F - SF = I,$$

or, after applying the right distributive law:

(22) $$(I - S)F = I.$$

To solve Equation (22) for F, the concept of a matrix inverse is needed. A matrix inverse is the analog of a reciprocal in ordinary multiplication. The inverse of matrix A is denoted A^{-1}, and has the property that $AA^{-1} = A^{-1}A = I$. Not every matrix has an inverse; whether a particular matrix has an inverse is related to whether a corresponding set of simultaneous equations has a unique solution. In the cases under consideration, the square matrix $(I - S)$ always has an inverse,[53] which will be denoted as $(I - S)^{-1}$. When each side of Equation (22) is left-multiplied by this matrix, the result is:

(23) $$F = (I - S)^{-1}.$$

The right side of Equation (23) can now be substituted for the summation in Equation (17):

(24) $$R = MFD = M(I - S)^{-1}D.$$

Here, in formula terms, is the solution sought: The vector R that shows how much each individual receives is expressed in terms of matrices M and S and vector D, all of which are based on given data. What remains is to show how to invert a matrix. This is a somewhat laborious procedure, but a personal computer can handle the details, since a spreadsheet program can invert as well as multiply matrices.[54]

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[53] As described in the text, the procedure for inverting a matrix is the same as the procedure for solving a set of simultaneous equations, where the jth coefficient of the ith equation is the element appearing in the ith row and jth column of the matrix. The elements of the matrix $(I - S)$ are the coefficients of the valuation equations discussed in Part III (p. 183). *Theorem 2* establishes that these equations always have a unique solution for a group of corporations that satisfy *Condition 2*. Hence, in these cases, the matrix $(I - S)$ is invertible.

[54] For the Macintosh: Microsoft Corporation, Microsoft Excel Function Reference 275–78 (ver. 4.0 1992); Lotus Development Corporation, Lotus 1-2-3 Users Guide 125–26 (1991). For IBM PC-compatible machines: Microsoft Corporation, Microsoft Excel User's Guide 177–79 (ver. 5.0 1993); Lotus Development Corporation, Lotus 1-2-3 User's Guide 212–13 (ver. 4.0 1993).

When done by hand, the procedure is the same as solving a set of simultaneous equations. This procedure, outlined in Part III above, involves a sequence of operations applied to a matrix which, in the end, transforms the matrix (or more precisely, the square part to the left of the vertical bar) into the identity matrix. This same sequence of operations will transform the identity matrix into the inverse of the original matrix.

To apply the formula in Equation (24) to the example in Figure 6, the three input matrices M, S and D, all of which have been given in the preceding pages, must be identified. Next, the matrix $I - S$ must be computed:

$$(25) \quad I - S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0.8 & 0.5 \\ 0.8 & 0 & 0.5 \\ 0.1 & 0.1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -0.8 & -0.5 \\ -0.8 & 1 & -0.5 \\ -0.1 & -0.1 & 1 \end{bmatrix}.$$

The next, and critical, step is inverting this matrix. As noted above, the technique is to apply the procedure used in Part III above to turn this matrix into the identity matrix, while applying the same steps to the identity matrix. These steps will transform the identity matrix into F, which is $(I - S)^{-1}$. Details are shown in Figure 7. Again, while the computation is intricate, the inversion could have been done by a computer rather than by hand.

FIGURE 7

$$\begin{bmatrix} 1 & -0.8 & -0.5 \\ -0.8 & 1 & -0.5 \\ -0.1 & -0.1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Starting position: F is on the left, and I is on the right.

$$\begin{bmatrix} 1 & -0.8 & -0.5 \\ 0 & 0.36 & -0.9 \\ 0 & -0.18 & 0.95 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0.8 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix}$$

Replace row 2 by the sum of itself and 0.8 times row 1 and
row 3 by the sum of itself and 0.1 times row 1.

$$\begin{bmatrix} 1 & -0.8 & -0.5 \\ 0 & 1 & -2.5 \\ 0 & -0.18 & 0.95 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2.222 & 2.778 & 0 \\ 0.1 & 0 & 1 \end{bmatrix}$$

Rescale row 2 by 0.36.

$$\begin{bmatrix} 1 & 0 & -2.5 \\ 0 & 1 & -2.5 \\ 0 & 0 & 0.5 \end{bmatrix} \qquad \begin{bmatrix} 2.778 & 2.222 & 0 \\ 2.222 & 2.778 & 0 \\ 0.5 & 0.5 & 1 \end{bmatrix}$$

Replace row 1 by the sum of itself and 0.8 times row 2 and
row 3 by the sum of itself and 0.18 times row 2.

$$\begin{bmatrix} 1 & 0 & -2.5 \\ 0 & 1 & -2.5 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2.778 & 2.222 & 0 \\ 2.222 & 2.778 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Rescale row 3 by 0.5.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 5.278 & 4.722 & 5 \\ 4.722 & 5.278 & 5 \\ 1 & 1 & 2 \end{bmatrix}$$

Replace row 1 by the sum of itself and 2.5 times row 3 and
row 2 by the sum of itself and 2.5 times row 3.

To derive the vector R, the matrix F must be left-multiplied by M,

$$(26)\ MF = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \end{bmatrix} \times \begin{bmatrix} 5.278 & 4.722 & 5 \\ 4.722 & 5.278 & 5 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0.528 & 0.472 & 0.5 \\ 0.472 & 0.528 & 0.5 \end{bmatrix}$$

and then right-multiplied by D,

(27)

$$R = MFD = \begin{bmatrix} 0.528 & 0.472 & 0.5 \\ 0.472 & 0.528 & 0.5 \end{bmatrix} \times \begin{bmatrix} 750 \\ 500 \\ 1{,}000 \end{bmatrix} =$$

$$\begin{bmatrix} 0.528 & \times & 750 & + & 0.472 & \times & 500 & + & 0.5 & \times & 1{,}000 \\ 0.472 & \times & 750 & + & 0.528 & \times & 500 & + & 0.5 & \times & 1{,}000 \end{bmatrix} = \begin{bmatrix} 1{,}132 \\ 1{,}118 \end{bmatrix}$$

The rightmost side of the bottom equation in Equation (27) is the vector R. Thus, if C_1 has $750 to distribute, C_2 has $500 and C_3 has $1,000, then of the full $2,250 of income, $1,132 is distributable to I_1, and $1,118 is distributable to I_2.

By now the destination has been overshot. The goal was to determine the percentage interest of each shareholder in each corporation. In determining the amount of income distributable to each shareholder, it was necessary to compute the percentage of income from each corporation that each shareholder has a right to receive. Assuming each corporation has only a single class of stock, a shareholder's percentage interest in a corporation can be measured by the percentage of the corporation's income that the shareholder has the right to receive when that income is fully distributed through the system. Looking closely at Equation (27), the $1,132 that I_1 is entitled to receive represents 52.8% of C_1's income of $750, plus 47.2% of C_2's income of $500, plus 50% of C_3's income of $1,000. This is just another way of saying that I_1 has a 52.8% interest in C_1, and so forth. The matrix P of percentage interests, with $P(I_i, C_j)$ in its ith row and jth column, shows the percentage interest of individual I_i in corporation C_j, and is given by the following formula:

(28) $P = MF.$

Since every corporation ultimately is wholly owned by individuals, the following should come as no surprise:

Theorem 3: The sum of all of the percentage interests in each corporation is equal to 100%:

$$\text{For all } j, \sum_k P(I_k, C_j) = 1.$$

The thesis here is that P states the properly computed percentage interest of each individual in each corporation taking into account the way the stock attribution rules should apply to strange loops. Nothing in the regulations under Sections 382 or 318 expressly mandates this way of treating strange loops. It is hard to see, however, how the regulations could be applied any differently and still yield coherent results. Moreover, since the regulations require attribution of stock held by a corporation to its shareholders on the basis of the value of their interests in the corporation, the percentage interest of an individual in a corporation, after applying these attribution rules, should represent the individual's share of the corporation's distributable income, which is what results from using these matrices to determine percentage ownership.

B.   *Value of the Loss Corporation*

After an ownership change, Section 382 limits the extent to which a loss corporation's pre-change net operating loss (and other) carryforwards can be used in any year to shelter that year's taxable income. The annual limitation is computed by multiplying the value of the loss corporation's stock on the date of the ownership change by an interest rate factor in effect at that time.[55] The interest rate to be used for this purpose is published monthly by the Service, based on current yields for long-term tax-exempt securities.[56]

If the loss corporation is part of a strange loop, there is an obvious danger of double counting if its entire outstanding stock is used

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[55]   I.R.C. § 382(b)(l).

[56]   I.R.C. § 382(f); Rev. Rul. 86-113, 1986-2 C.B. 59.

to measure value. Since directly self-owned (that is, treasury) stock is ignored in measuring the value of the loss corporation, presumably indirectly self-owned stock should be ignored as well. For the two corporation example in Figure 1, Equations (5) and (6) give a value of $1,474 for the stock of C_1 and $1,526 for C_2, even though the two corporations together have only $300 worth of underlying assets. Clearly, some adjustment is necessary to eliminate the spurious value created by the cross ownership.

This problem is not limited to strange loops. Consider the simplest untangled hierarchy: Parent P owns all of the stock of subsidiary S. Both have net operating loss carryforwards; S owns an asset worth $1,000; P owns nothing but the stock of S. The shareholders of P sell all their stock to unrelated parties for $1,000, causing both P and S to have an ownership change. Suppose the published long-term tax-exempt rate is 6%. The value of P's stock is $1,000; the value of S's stock is also $1,000. Absent further adjustments, P can use $60 (6% of $1,000) of its loss carryforwards each year after the sale; likewise, S can use $60 of its loss carryforwards. The double counting arises because the value of P is attributable solely to the value of S,[57] so the value of S is used to support the use of P's loss carryforwards as well as S's own carryforwards.

The Service has issued two sets of proposed regulations that deal with overlapping value. One set applies to consolidated groups; the other applies to nonconsolidated affiliates with at least 50% common ownership. For consolidated groups, these regulations cause the loss carryforwards of the entire group to be limited if the common parent

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[57] This simple example makes the artificial assumption that the loss carryforwards add nothing to the value of P and S. In reality, loss carryforwards add a premium to value even when limited by Section 382. The use of a tax-exempt interest rate, rather than a taxable rate, to compute the Section 382 limitation is intended to offset this premium in a rough way and thereby discourage "trafficking" in corporations with loss carryforwards. H.R. Rep. No. 99-841, at II-188 (Conf. Rep. 1986), *reprinted in* 1986-3 C.B. (vol. 4) 188.

has an ownership change,[58] even though a less than wholly-owned consolidated subsidiary might not have an ownership change if viewed separately.[59] The consolidated net operating loss of the group is limited by a consolidated group Section 382 limitation. This limitation is computed by multiplying the applicable tax-exempt rate by the value of the group as a whole, which takes into account only the value of stock of members of the group held by nonmembers.[60] In the example in the preceding paragraph, the consolidated group limitation of the group would be $60, based on the $1,000 of the value of *P* stock held by nonmembers. The value of the *S* stock would be ignored, since the stock is held by group member *P*.

The proposed regulations that measure the value of consolidated groups with an ownership change are issued under the Service's general authority to prescribe the tax treatment of corporations electing to file consolidated returns.[61] The proposed regulations for nonconsolidated affiliates are issued under Section 382 itself, which authorizes the Service to issue regulations providing, in the case of a controlled group, appropriate adjustments to value so that items are not taken into account more than once.[62] For this purpose, the term "controlled group" is used as defined in Section 1563(a), with a 50% rather than an 80% threshold of common ownership.

The regulations for controlled but nonconsolidated groups eliminate overlapping value by subtracting from the value of each member the value of any stock it directly owns in another member.[63] Thus, if the corporations in the preceding example did not file consolidated

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[58]   Prop. Treas. Reg. § 1.1502-92(b)(l), 56 Fed. Reg. 4194 (Feb. 4, 1991).

[59]   If *P* owned 80% of *S*, and 60% of the stock of *P* changed hands, then *P* would have an ownership change, but *S*, viewed separately, would not, because the new owners of the *P* stock would own indirectly only 48% of *S*.

[60]   Prop. Treas. Reg. § 1.1502-93(b), 56 Fed. Reg. 4194 (Feb. 4, 1991).

[61]   I.R.C. § 1503(a).

[62]   I.R.C. § 382(m)(5).

[63]   Prop. Treas. Reg. § 1.382-5(c)(l), 56 Fed. Reg. 4183 (Jan. 29, 1991).

returns, *S* would have a value of $1,000, but *P* would have a value of zero. Like the consolidated group case, double counting is avoided, but here it is the value of *P* that is cut back rather than the value of *S*. The affiliates are likely to care where the cut back in value is made because some members may have more loss carryforwards, and a greater capacity to use them, than others. The regulations accommodate this concern by permitting members of a controlled group, within limits, to elect to restore value to other members that hold their stock, with a corresponding reduction in the value of the electing members' stock.[64] In the simple case, *S* could elect to restore all or part of its value to *P*, which would make sense if, for example, only *P* had loss carryforwards.

The right of one member of a controlled group to restore value to another is subject to two restrictions. First, the amount restored cannot exceed the unadjusted value of the stock of the restoring member held by the receiving member, since this value is what was excluded from the value of the receiving member before the restoration. Thus, a member cannot restore more than the amount taken away. Second, the amount restored cannot exceed the value of the stock of the restoring member held by the receiving member, computed after applying the initial adjustment but before any restorations, plus a pass through of any amounts restored by other members to the restoring member. This second restriction limits restorations to amounts attributable to assets other than the stock of members.[65]

A tangled hierarchy can be subject to either set of regulations, depending on which members are included in a group filing consolidated returns. The effect of strange loops on the standards for consolidation is itself a point that requires discussion, which is deferred to Part V. Returning to the two-corporation example in

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[64]   Prop. Treas. Reg. § 1.382-5(c)(2), 56 Fed. Reg. 4183 (Jan. 29, 1991).

[65]   This restriction is illustrated for a hierarchy without strange loops in Prop. Treas. Reg. § 1.382-5(g), Ex. (3), 56 Fed. Reg. 4183 (Jan. 29, 1991).

Figure 1, assume that they do file consolidated returns. In that case, any ownership change would be determined for the group as a whole, and its consolidated Section 382 limitation would be based on the $300 of total value of the C_1 and C_2 stock held by nonmembers (that is, I_1 and I_2). This result under the consolidated return regulations sensibly avoids any duplication in value.

Now assume that C_1 and C_2 do not file consolidated returns, either because they are not eligible or because they do not so elect. In that case, if C_1 has an ownership change, the value of C_1's stock is determined initially by subtracting from the gross value of $1,474 the value of C_1's interest in C_2, which is 90% of $1,526, or $1,374. The net amount of $100 is simply $A(C_1)$, which is the value of C_1's assets apart from its interest in C_2. The parties can improve on this result if C_2 is willing to restore all or part of its value to C_1. The amount that can be restored is limited to the lesser of (1) the value of C_1's interest in C_2, or $1,374 or (2) the reduced value of C_1's interest in C_2 (which is 90% of $A(C_2)$, or $180), plus any amount restored from C_1 to C_2 (assumed to be zero). If C_2 restores the maximum amount of $180 to C_1, then C_1's value can be raised from $100 to $280 for purposes of computing its Section 382 limitation. This restoration would make sense if C_2 had no loss carryforwards (and therefore did not care about losing value for Section 382 purposes). Even if C_2 had loss carryforwards, it might be willing to restore value to C_1, because without consolidation, the sale by I_1 of her stock would not cause C_2 to have an ownership change.

Suppose C_2 did restore $180 of value to C_1, and then, because of a subsequent sale by I_2, C_2 had an ownership change. It is tempting to suppose that, in computing its Section 382 limitation, C_2 could include in its value not only its full $200 of assets other than the stock of C_1, but also the maximum $90 of value that C_1 would be permitted to restore to C_2. If permitted, most of the asset value of these two corporations would be doing double duty, contributing to the valuations for the Section 382 limitations of both. The proposed regulations

appear to foreclose this possibility by requiring that "appropriate adjustments" be made to prevent any duplication of value, including any value already used to determine a limitation under Section 382 with respect to losses of a controlled group member from the same period.[66]

One might wonder whether in some perverse way the cross ownership between C_1 and C_2 could be exploited to inflate the Section 382 limitation. In our example, it is limitation (2) above, based on reduced values, that is restrictive. This limitation on what C_2 could restore to C_1 is based, among other things, on how much C_1 restores to C_2. There is an apparent circularity here: In which direction is the limitation on restorations to be computed first? One answer is to apply both limitations simultaneously. Let ρ_1 be the amount restored by C_1 to C_2, and let ρ_2 be the amount restored the other way. Limitation (2), applied bidirectionally, can be expressed as follows:

$$(29) \qquad \begin{aligned} \rho_1 &\leq S(C_2, C_1) A(C_1) + \rho_2, \\ \rho_2 &\leq S(C_1, C_2) A(C_2) + \rho_1, \end{aligned}$$

One of these equations will always be trivially true; which one it is depends on whether ρ_1 or ρ_2 is bigger. For example, if ρ_2 is bigger, then only the second of these equations imposes a meaningful restriction. Even so, the second equation imposes no restriction on the absolute value of ρ_2; it only requires that ρ_2 not exceed ρ_1 by more than C_1's share of C_2's asset value.

In a two-corporation case, even this mild restriction is strong enough: Increasing ρ_2 in order to enhance C_1's Section 382 limitation is pointless if the cost of doing so is to increase ρ_1, because the regulations reduce the value of a controlled group member by the amount that it restores to others. The real opportunities exist, if at all, in controlled groups with multiple strange loops. Suppose, for example, that C_2 is also part of a strange loop with another corporation C_3 (Figure 6 shows one way this might occur). Circular restorations of value be-

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[66]   Prop. Treas. Reg. § 1.382-5(c)(4), 56 Fed. Reg. 4183 (Jan. 29, 1991).

tween C_2 and C_3 might have no net effect on the values of these two corporations for Section 382 purposes, but could increase the amounts that each could restore to C_1. The regulations apparently forestall even this possibility, however. In a rare reference to strange loops, the regulations require adjustments to "take into account" cross ownership of stock by controlled group members.[67] While neither the examples in the regulations nor the preamble provide any further guidance as to what these adjustments might be, measures to restrict inflation in value by circular restorations surely must be among them.

In their zeal to prevent duplications in value, the proposed regulations overlook the potential for members of a controlled group to suffer duplications in income. If a controlled group member pays dividends to another, and the two corporations are not members of an affiliated group meeting the 80% ownership thresholds for consolidation, then at least 20% of the dividend is taxable.[68] In such a case, arguably only 80% of the value of the intragroup stock holdings should be excluded in determining the Section 382 limitation of each member. Since the other 20% can be considered to generate taxable income, the holder should be permitted to apply its loss carryforwards against an assumed return on the value of that 20% of the intragroup holding.

Conversely, to the extent the dividends received deduction applies to holdings outside of controlled groups with 50% common ownership, the proposed regulations, and the statute that authorizes them, arguably do not go far enough in restricting duplications in value. If, however, two corporations are only loosely related, an ownership change of one is far less likely to coincide with an ownership change of the other. There is less likelihood, therefore, that the same value

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[67]   Prop. Treas. Reg. § 1.382-5(c)(4)(ii), 56 Fed. Reg. 4183 (Jan. 29, 1991).

[68]   I.R.C. § 243. Unless the recipient of the dividend directly owns 20% by vote and value of the stock of the payor, 30% of the dividend will be taxable.
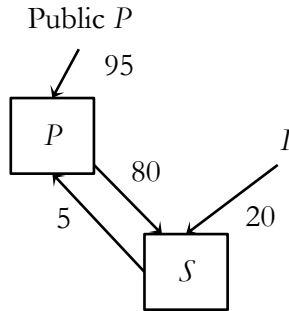
would be taken into account in determining the Section 382 limitation for more than one corporation.

V. Affiliated Groups

A. *Subsidiary in a Strange Loop*

Suppose parent *P* owns 80% of the stock of subsidiary *S*, and individual *I* owns the other 20%. *S* buys 5% of the stock of *P* for cash, either from *P*'s shareholders or from *P* itself, as shown below.

FIGURE 8

Public *P*

95

P

80

5

I

20

S

This transaction may be a Section 304 dividend, but that is not the question here. Rather, the question is, can *P* and *S* still consolidate?

A plausible first reaction is yes, consolidation should be allowed. In the two corporation case, consolidation is allowed if the parent directly owns 80% of the voting power and value of the subsidiary's stock.[69] On the face of it, that requirement is satisfied here.

In some sense, however, *P*'s ownership of *S* has dropped below 80%. *S* now owns a 4% (5% of 80%) interest in itself through *P*. If this interest is treated like treasury stock, and therefore disregarded, then *P*'s interest rises to 20/96ths, or 20.83%. Consistent with this view, *P*'s real interest in *S* must be limited to 79.17%, which is insufficient to justify consolidation.

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[69] I.R.C. § 1504(a).

Perhaps this proves too much. If indirect interests were to be taken into account for consolidation, what about the indirect interests in *S* held by the other shareholders of *P*? In any corporate tree without strange loops, the shareholders of the common parent hold indirect interests in subsidiaries that exceed 20%; but that is beside the point. It is direct links that count, because affiliated group status is a measure of affinity through ties of share ownership. In Figure 8, to the extent that *P*'s interest in *S* exceeds 20%, it is because of a chain of indirect ownership that runs through *P*. Should this defeat consolidation any more than if *I* owned some *P* stock directly? Before answering this question, it is necessary to look more closely at how the consolidation rules apply to strange loops when based on direct links only.

B.   *Paradoxes with Direct Links*

Section 1504 establishes two requirements for an affiliated group. First, the common parent must own directly 80% (by vote and value) of the stock of at least one other member. Second, 80% of the stock of each member, except the common parent, must be owned directly by other members.[70] Thus, the literal terms of the statute refer only to direct links. Yet, when strange loops are considered, paradoxes abound.

1.   *Two Common Parents*

Consider the example back in Figure 1, with 90% cross ownership. C_1 directly owns 90% of the stock of C_2, so the two corporations should be permitted to consolidate, with C_1 as the common parent. Yet, by the same reasoning, C_2 also qualifies as the common parent, because it owns 90% of C_1.

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[70]   I.R.C. § 1504(a).

A consolidated group with two common parents is not a logical impossibility, although the regulations clearly contemplate only one. For example, the common parent acts as agent for all of the subsidiaries in dealings with the Service, including filing returns, making tax elections and making claims for refund on behalf of the group.[71] One, however, could deal with the need for a single agent in the odd case of a group with multiple common parents by requiring the group to name one of them to act as agent, just as a partnership subject to the unified audit rules is required to have a tax matters partner.[72]

A more troubling question arises when one of the corporations, but not the other, has an ownership change when viewed separately. This would occur, for example, if only I_1's stock changed hands. The existence of an ownership change for the group, and hence for C_2, would depend on whether C_1 or C_2 is the common parent.[73] Here, the choice of common parent has more than merely administrative consequences.

2. *No Common Parent*

A more troubling case is shown in Figure 9. C_1 clearly owns, directly and indirectly, all of the outstanding stock of C_2 and C_3. Yet, just as clearly, these three corporations do not form an affiliated group within the literal terms of Section 1504(a). To be an affiliated group, one of the corporations must, as common parent, directly own 80% of the voting power and value of at least one of the other corporations. Here, no corporation directly owns more than 50% of the outstanding stock of any other corporation. Consequently, not only is this group ineligible to file consolidated returns, but its members cannot claim the 100% dividends received deduction, which is available
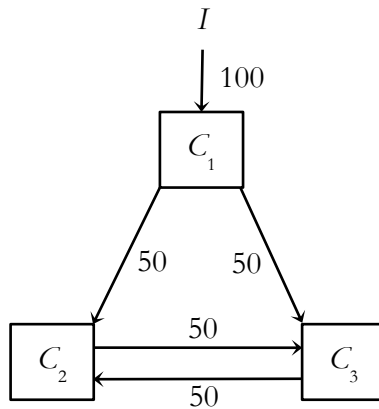
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[71]   Treas. Reg. § 1.1502-77(a).

[72]   I.R.C. § 6231(a)(7).

[73]   *See supra* note 58.

only for members of affiliated groups.[74] The tax consequences of losing the 100% dividends received deduction can be quite serious, because of the way dividends can cycle repeatedly through the strange loops. Part VI below quantifies this effect.

FIGURE 9

A further oddity of the group shown in Figure 9 is that interposing a holding company between I and C_1 eliminates the problem: The new holding company qualifies as the common parent of an affiliated group, since it owns 100% of C_1, and 100% of the stock of the three corporations other than the holding company is owned by other group members. Further weird consequences of adding holding companies are discussed in Part V.B.4.

3. *Wrong Common Parent*

The group of corporations in Figure 10 does form an affiliated group, because at least 80% of the stock of each member is directly owned by other members, and one of the members, C_2, directly owns 80% of the stock of one of the others.

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[74]   I.R.C. § 243(a)(3), (b).

FIGURE 10
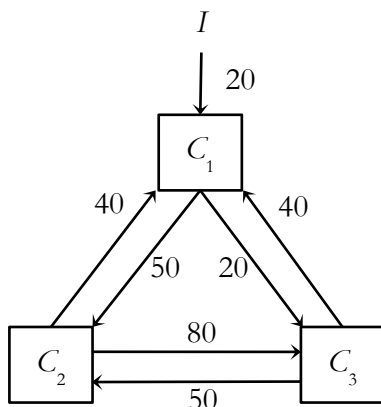
Yet, C_2 is a peculiar choice of common parent. For one thing, it is 100% owned by its subsidiaries. Indeed, the only member with any outside ownership is C_1, which is purportedly a subsidiary. Anyone not familiar with the literal terms of Section 1504(a) would have assumed that C_1 was the common parent.

4. *Oddball Accretion*

The sequence of diagrams in Figure 11 confounds an attempt to apply the current law definition of an affiliated group, and perhaps any concept of an affiliated group based solely on direct links, to corporations with strange loops.

Figure 11A shows a variation of the initial two corporation example shown in Figure 1. The only difference is that I_1's interest in C_1 is held through a holding company, C_3, rather than directly. Although C_1 and C_2 can form an affiliated group, with either as the common parent, C_3 cannot be a member of this affiliated group because it is not 80% owned by other group members, and it cannot be the common parent because it does not directly own 80% of the stock of any other member.

FIGURE 11
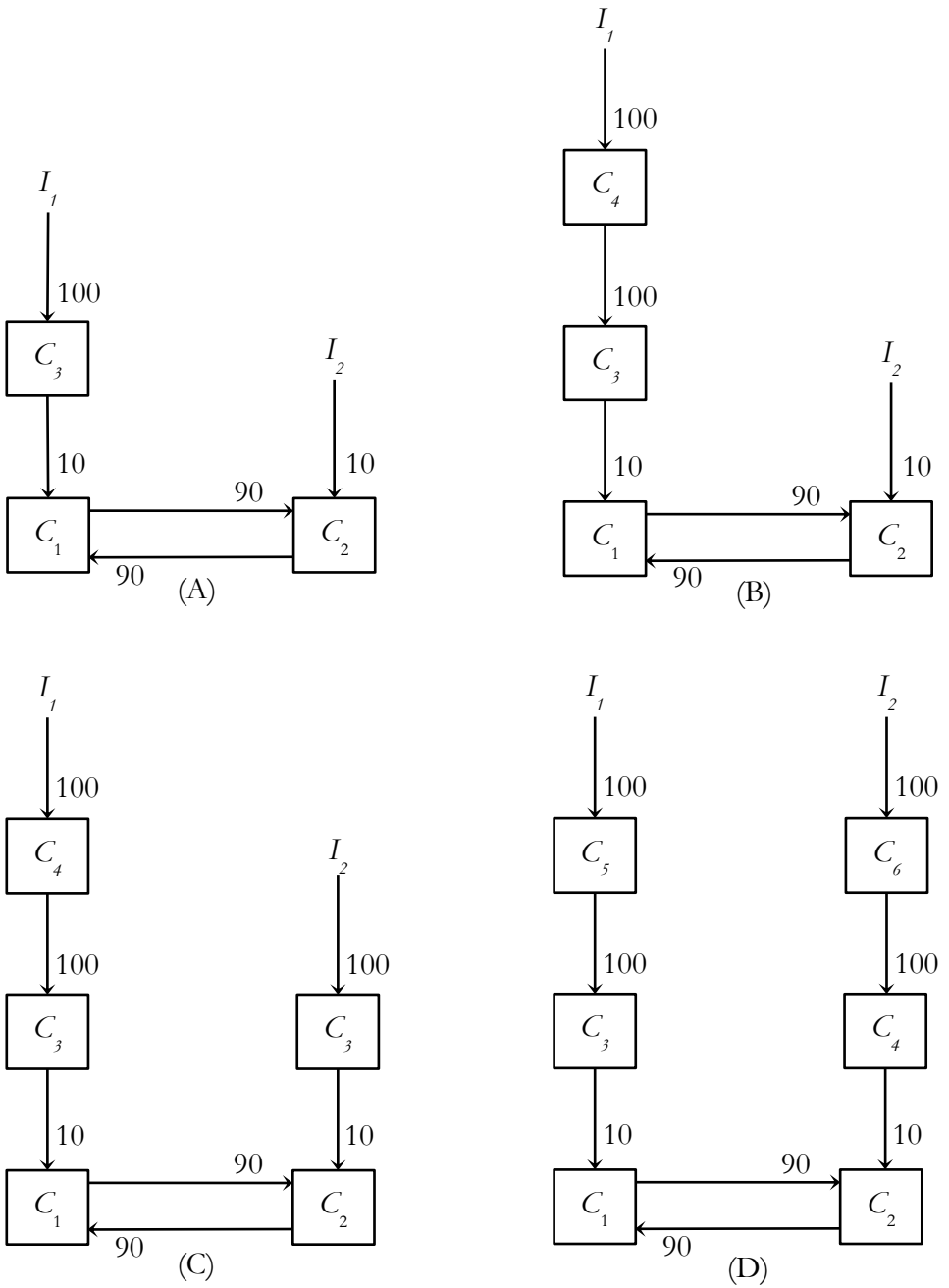
Now suppose that I_1's interest in C_3 is held through another holding company, C_4, as shown in Figure 11B. This change causes all of the corporations to qualify for inclusion in a single affiliated group, because C_4 as the common parent owns more than 80% of C_3, and each of the other three corporations is at least 80% owned by other group members. The effect of interposing C_4 is to "attach" it and C_3 to the group in Figure 11A consisting of C_1 and C_2.

Figure 11C adds a holding company on the other side of the diagram: I_2's interest in C_2 is held through a holding company, C_5. The four corporations that formed a group in Figure 11B continue to form a group here, but this group does not include C_5, which is not owned at all by other group members and cannot qualify as a common parent since it only owns 10% of C_2 and no stock of any other member.

In Figure 11D the picture becomes more bizarre. Two inconsistent affiliated groups are possible: one that includes C_1, C_2, C_3 and C_4, with C_4 as the common parent and another that includes C_1, C_2, C_5 and C_6, with C_6 as the common parent. One might consider an all-inclusive group covering the six corporations; after all, the possibility of more than one common parent already has been addressed. What is new here, however, is that both C_4 and C_6 can qualify only as common parents, since their stock is entirely owned by separate nongroup members. Their joint inclusion would stretch the affiliated group concept to include corporations with no overlap in ultimate beneficial ownership. The folly of this approach becomes apparent when C_4 and C_6 are each imagined to be sizable corporations with diverse enterprises, with C_1, C_2, C_3 and C_5 relatively insignificant by comparison. The presence of this modest link would enable the two common parents to reap all of the benefits of consolidation, including offsetting profits of one with losses of the other. This result might be welcome to

fans of safe harbor leasing,[75] but surely is not intended by the consolidated return rules.

C. *Towards Coherence*

Can there be a coherent concept of affiliation where strange loops are involved? Clearly the current statutory scheme, which is based purely on direct links, leaves much to be desired on this score. The oddities presented by Figure 11 originate with C_4, which caused C_3 to be grafted onto the group consisting of C_1 and C_2. There is something troubling about this linkage, since C_3 has only a 52.6% interest in C_1 when the strange loop is taken into account, with the remaining 47.4% interest owned ultimately by I_2, who is completely unrelated to C_3.

These considerations suggest that there is something special about strange loops requiring that the indirect interests they reflect be taken into account in determining affiliated group status. At the same time, affiliation is a relationship that depends on the degree of affinity between corporations, not ultimate ownership. Consider, therefore, the following measure of affinity: Two corporations are linked if at least 80% of the income of one flows through the other, when the income is fully flushed through the system by an infinite series of distributions. I propose that strange loops can be handled in a coher-

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[75] I.R.C. § 168(f)(8) (before amendment in 1982), added by the Economic Recovery Tax Act of 1981, Pub. L. No. 97-34, § 201, 95 Stat. 172, 203. This provision authorized fictitious leases, given effect for tax purposes only, that allowed taxpayers with net operating losses to transfer the benefits of cost recovery deductions and investment credits on new equipment to other taxpayers who could use these tax benefits to shelter their taxable income. The intended purpose of these rules was to ensure that the investment incentives created by these tax benefits were available to taxpayers with no current taxable income. The practice quickly ran into political hot water, and was repealed a year later by the Tax Equity and Fiscal Responsibility Act of 1982, Pub. L. No. 97-248, § 209, 96 Stat. 324, 442.

ent manner if the term "affiliated group" is defined to be a group of corporations linked in this manner.

First, a more precise concept of "flow" is necessary, since this governs linkage. Let $F(C_i, C_j)$ represent the degree of FLOW from corporation C_j to corporation C_i. It turns out that this amount appears in the ith row and jth column of the matrix F used in Part IV to compute percentage interests. From the derivation of the percentage interest matrix P in Equation (28) and the definition of matrix multiplication, the result is:

$$(30) \qquad P(I_k, C_j) = \sum_i M(I_k, C_i) F(C_i, C_j).$$

Equation (30) states that the percentage of corporation C_j's income that is ultimately distributable to individual I_k can be determined by adding up I_k's direct percentage interest in the flow of income from each corporation that has income flowing from C_j. For example, Equation (26) shows that I_1's 47.2% interest in C_2 arises by owning 10% of C_1, through which C_2's income flows 4.722 times on its way to the individual shareholders. A corporation's income always flows through itself at least once, so $F(C_i, C_i)$ is always at least equal to one, and if C_i is not part of any strange loop, then $F(C_i, C_i)$ will be equal to one. C_i is LINKED to C_j if either $F(C_i, C_j) \geq 0.80$ or $F(C_j, C_i) \geq 0.80$.

A subset $\{G\}$ of $\{C\}$ is an AFFILIATED GROUP if each of the following is true:

(1)   each member of $\{G\}$ is linked to at least one other member of $\{G\}$;

(2)   no corporation outside of $\{G\}$ is linked to a member of $\{G\}$; and

(3)   for any proper subset of $\{G\}$, at least one member of the subset is linked to a member of $\{G\}$ outside the subset.

The first condition requires a chain of linkages; the second requires that the chain be as all-inclusive as possible, and the third ensures that the whole of $\{G\}$ is properly linked together.

FIGURE 12

Flow Matrix

	C_1	C_2	C_3	C_4	C_5	C_6
C_1	5.26	4.74	0	0	0	0
C_2	4.74	5.26	0	0	0	0
C_3	0.526	0.474	1	0	0	0
C_4	0.526	0.474	1	1	0	0
C_5	0.474	0.526	0	0	1	0
C_6	0.474	0.526	0	0	1	1

(A)

(B)

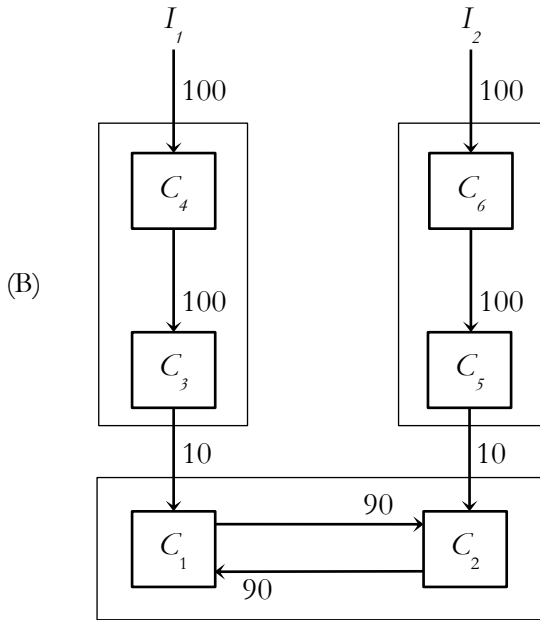Figure 12A shows the flow matrix F for the group of corpora-
tions in Figure 11D. A linkage occurs between two corporations C_i
and C_j only if the ijth entry or the jith entry in F is at least equal to
0.80. Instead of a single affiliated group, there are three distinct
groups, as shown by the boxes superimposed over the matrix and the
corporate diagram in Figure 12B. This outcome accords with a com-

mon sense view of which corporations should be regarded as affiliated.

A few properties of this affiliation concept can be noted. First, no corporation can be a member of more than one affiliated group, since its linkages with members of two distinct groups would itself cause the two groups to fuse. Second, each member must at least have a link with one other particular member; there is no analogue to the concept under the current law definition that allows the 80% ownership test to be satisfied by the holdings of several other members at once. Third, there is no concept here of a common parent, which is an unavoidable casualty of the accommodation of strange loops. Consequently, if this concept of affiliation were to be adopted, the rules that single out the common parent for special treatment would need to be revised.

This concept of affiliation is not coextensive with the current law definition even in the absence of strange loops. For example, in Figure 13A all four corporations would be consolidated under current law, but C_4 would be left out under the proposed definition. Conversely, in Figure 13B, under current law there would be two groups, one consisting of C_1 and C_3, and the other consisting of C_2 and C_4. Under the proposed definition, all four corporations would be included in a single group.

Figure 13B illustrates a slight overinclusiveness: C_2 is allowed to consolidate with C_1 under the proposed rule even though there is only a 75% link between them, merely because they share ownership interests in C_4. This is hardly more significant than a similar overinclusiveness that arises under the current law definition, where an unrelated investor can have a greater than 20% interest in a second-tier subsidiary by holding interests in a first-tier subsidiary as well. Moreover, C_1's interest in C_2 could not drop below 75% without destroying their affiliated status, since C_4 no longer would have the requisite linkage with C_1.

FIGURE 13
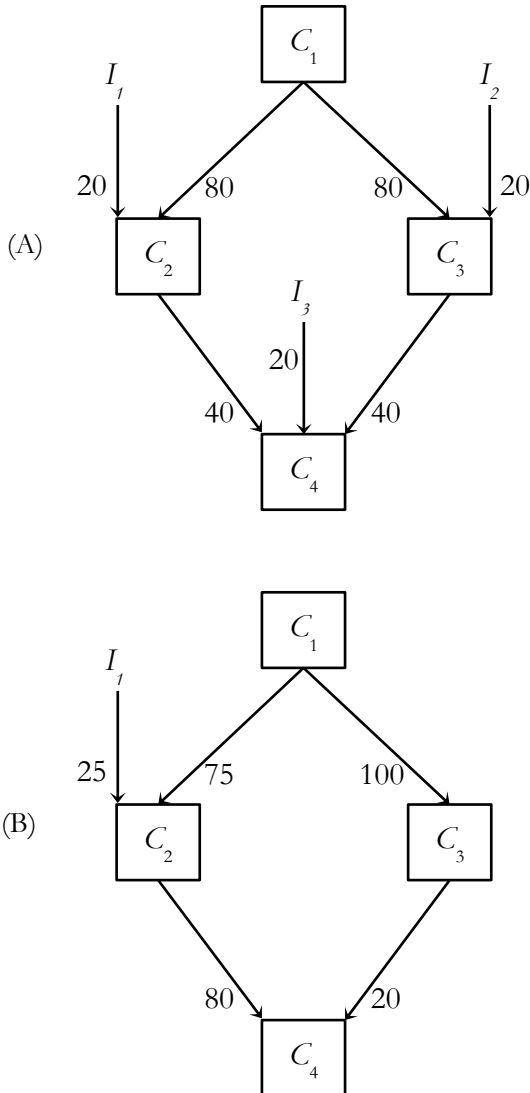
This approach would preserve the consolidation of the two cor-
porations shown in Figure 8. In that case $F(P,S) = 0.80/(1 - 0.80 \times 0.05) = 0.833$, which exceeds the requisite 80% threshold of linkage.

There is, perhaps, no perfect definition of an affiliated group,
since any measure of affinity between corporations is unlikely to be

transitive: If C_1 and C_2 are closely related, and C_2 and C_3 are closely related, it is not necessarily true that C_1 and C_3 are closely related to the same degree. Yet it is possible to craft a definition that avoids the paradoxes of current law when applied to corporate structures with strange loops. The current law definition has no offsetting advantage except, perhaps, familiarity.

VI. Tax Burden on Cycled Dividends

To the extent that corporations forming a strange loop are included in an affiliated group, the cycling of dividends through the loop has no federal income tax cost.[76] If, however, the corporations are not part of a single affiliated group, only a partial dividends received deduction is allowed. If the recipient directly owns at least 20% of the payor's stock, the recipient can deduct 80% of the dividends received;[77] otherwise, the deductible percentage is 70%.[78]

It is possible to quantify the tax cost of a strange loop, which is a function of the effective tax rates imposed on dividends passing between each link. If income originating in a corporation is distributed all the way upstream to the ultimate shareholders, what percentage ends up in the hands of the government?

Before analyzing the details of the general case, it is worth considering how general it is. There can be any number of corporations, with any degree of interlocking stock ownership among any pair of them. A separate tax rate can be applied to the dividends flowing from any particular corporation to each other corporation, which might depend on the degree of stock ownership, the nationality of either corporation or other factors. There can be any number of strange loops, and each can have any number of links. A simplifying assumption that will be maintained is that each corporation has only a single class of stock. Also, each corporation is assumed to have enough earnings and profits to cause its distributions to be treated as dividends for tax purposes.

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[76] There might, however, be a state income tax cost.

[77] I.R.C. § 243(c).

[78] I.R.C. § 243(a)(l).

A. *Government as a Phantom Shareholder*

Each corporation generates some cash to distribute. With this information, together with charts of share ownership and intercompany tax rates, the task is to compute how much ultimately will be distributed to each shareholder, and how much will be lost in taxes. The matrix mechanics used in Part IV above to determine percentage interests can be modified slightly to determine effective tax burdens.

The key is to treat the government as an additional "phantom" individual shareholder I_G, receiving its share of each round of dividends in accordance with the applicable tax rates.

Let $T(C_i, C_j)$ be the TAX RATE on dividends paid by C_j to C_i. (Taxes on dividends to individual shareholders are ignored here, since only the tax burden on intercorporate dividends needs to be measured.) C_i's after-tax share of a distribution paid by C_j is denoted $S'(C_i, C_j)$, and is given by the following formula:

$$(31) \qquad S'(C_i, C_j) = (1 - T(C_i, C_j)) S((C_i, C_j)).$$

Consider the three corporations in Figure 9, which under the current law definition would not be permitted to consolidate. Dividends among them would benefit only from the 80% dividends received deduction, so with a 35% tax rate, there still would be a 7% tax, and only 93% of the amount received would be available for the recipient to pay to its shareholders. In this case, the matrices S and S' would be:

$$(32) \qquad S = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 \\ 0 & 0.5 & 0 \end{bmatrix}; \quad S' = \begin{bmatrix} 0 & 0.465 & 0.465 \\ 0 & 0 & 0.465 \\ 0 & 0.465 & 0 \end{bmatrix}.$$

So that Condition 1 remains satisfied, the portion of the intercorporate dividend that disappears in taxes should be treated as distributable to the phantom shareholder I_G. Consequently, the matrix M showing each individual's percentage share of each corporate dividend needs to be augmented by adding another row showing the portion of each corporation's dividend lost to intercorporate taxes;

call the augmented matrix M'. The element in column j of the bottom row of M' shows the portion of C_j's dividend that is paid in taxes:

$$(33) \qquad m'_{G,j} = \sum_i T(C_i, C_j) S(C_i, C_j).$$

The matrix M' has two rows when the phantom shareholder is added, and three columns:

$$(34) \qquad M' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.07 & 0.07 \end{bmatrix}.$$

A modified flow matrix F' showing AFTER-TAX FLOW can be computed by inverting $(I - S')$. The calculation having been left to a computer, the resulting matrix is:

$$(35) \qquad F' = (I - S')^{-1} = \begin{bmatrix} 1 & 0.869 & 0.869 \\ 0 & 1.276 & 0.593 \\ 0 & 0.593 & 1.276 \end{bmatrix}.$$

Left-multiplying this matrix by M' gives a matrix P', which shows the percentage of each corporation's distributions that reaches each individual shareholder, together with the percentage that is paid in taxes:

$$(36) \qquad P' = M'F' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.07 & 0.07 \end{bmatrix} \times \begin{bmatrix} 1 & 0.869 & 0.869 \\ 0 & 1.276 & 0.593 \\ 0 & 0.593 & 1.276 \end{bmatrix} = \begin{bmatrix} 1 & 0.869 & 0.869 \\ 0 & 0.131 & 0.131 \end{bmatrix}.$$

This last matrix shows that none of C_1's distributed income is subject to corporate tax, which is to be expected since C_1 is wholly owned by I_1. By contrast, the income of C_2 and C_3, when fully distributed is subject to cumulative corporate taxes equal to 13% of the income, and only 86.9% will reach I_1. The cycling here nearly doubles the stated 7% rate of intercorporate tax.

B. *E&P and Basis Adjustments*

Under the consolidated return regulations, whenever a subsidiary accumulates E&P during a taxable year, each other member of the group holding stock of the subsidiary must adjust its basis in the subsidiary's stock by its share of these earnings and profits.[79] This basis adjustment itself generates earnings and profits of that member in an equal amount.[80] These earnings and profits, in turn, trigger a basis adjustment of the stock of that member held by other members of the group. In an ordinary corporate tree, these effects ripple upwards from the twigs to the trunk: Adjustments are made first to higher tier subsidiaries, then to lower tier subsidiaries.

Needless to say, the Service concocted these rules without strange loops in mind. If two subsidiaries own stock in each other, there is no clear way to distinguish the higher tier from the lower tier subsidiary, so the sequence of adjustments is also unclear. As it happens, however, these adjustments can be made for corporations in a strange loop without having to decide on an ordering rule. In a two-corporation strange loop, the cumulative E&P adjustments can be added up in the same manner as cycled dividends. Indeed, this procedure can be generalized for any number of corporations, and the total adjustment for corporation C_i resulting from the pre-adjustment E&P of C_i (as a percentage of its own E&P) is $F(C_i, C_i)$, which can be read directly off the flow matrix F. Of course, a corporation in a strange loop will make repeated adjustments in respect of its own E&P; that $F(C_i, C_i)$ for such a corporation C_i is always greater than 100% reflects this fact.

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[79] Treas. Reg. § 1.1502-32(a), (b)(l)(i). New regulations have been proposed that would substantially revise these investment adjustment rules; the principal change is to determine adjustments to subsidiary stock basis by reference to its taxable income rather than its E&P. Prop. Treas. Reg. § 1.1502-32(b)(3)(i), 57 Fed. Reg. 53,634 (Nov. 12, 1992). These adjustments "tier up" through the consolidated group in much the same manner as under the existing regulations and apply similarly to strange loops.

[80] Treas. Reg. § 1.1502-33(c)(4)(ii)(a).

If vector D is assigned the values of each corporation's pre-adjustment E&P, then the post-adjustment E&P is given by the vector FD. The ith row of this vector, computed, using the formula for matrix multiplication, is:

$$(37) \qquad\qquad FD(C_i) = \sum_j F(C_i, C_j) D(C_j).$$

The basis adjustments track the E&P adjustments, except that a group member gets no basis adjustment in its own stock.

Similar cycling occurs when E&P are negative. If a subsidiary has a E&P deficit for the year, the consolidated return regulations provide for a downward adjustment in the basis of the stock held by each other member of the group.[81] This basis adjustment itself decreases the E&P of each such other member,[82] triggering a further round of basis adjustments.

The foregoing assumes that all members in the strange loop are subsidiaries. If the common parent is in the loop, it breaks the chain of adjustments. The regulations provide for adjustments only in respect of the E&P of subsidiaries, which means each member of the group other than the common parent.[83] Presumably the drafters of the regulations assumed that stock of the common parent would be held only by nonmembers, which is always the case in the absence of strange loops.

Whenever the flow from one member to another, as shown in the flow matrix, exceeds one, the basis adjustment exceeds 100% of the other member's share of the E&P of the first member.[84] At first glance, this multiplier effect on stock basis might seem to create an opportunity to shelter gains on the sale of a corporation in a strange loop. While some such opportunities may exist, the picture is consid-

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[81]   Treas. Reg. § 1.1502-32(b)(2)(i).

[82]   Treas. Reg. § 1.1502-33(c)(4)(ii)(a).

[83]   Treas. Reg. § 1.1502-l(c).

[84]   *See* Walter, *supra* note 16, at 913.

erably more complicated, because a strange loop has a multiplier effect on value as well as basis.

Earnings and profits that are distributed currently generate no basis adjustments. The distributions carry E&P with them as they cycle through any strange loops, and a full (that is, infinitely repeated) distribution carries out all of the E&P. If E&P accumulated in a prior year is subsequently distributed, the earlier series of adjustments reverses itself: A distribution of previously accumulated E&P generates a negative basis adjustment[85] each time it cycles through the strange loop.

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[85]   Treas. Reg. § 1.1502-32(b)(2)(iii)(a).

VII. FORMING A STRANGE LOOP

When a corporation's stock becomes part of a strange loop, it becomes, to some extent, indirectly self-owned. To that extent, it resembles treasury stock. Yet, the tax law, by and large, treats such stock just like any other asset of the holder. Its acquisition gives the buyer a cost basis, and, except when Section 304 applies, the transaction is an ordinary sale to the seller, not a redemption.

Section 304 recognizes that when a subsidiary acquires stock of its parent, the transaction is similar enough to a redemption to justify treating it as such in order to prevent the seller from bailing out corporate earnings at capital gains rates. Yet, historically, the bailout opportunity has not been limited to the formation of strange loops: Section 304 applies equally to "brother-sister" acquisitions by one corporation of another under common control with it, even though no strange loop is formed.

Bailout isn't what it used to be: The Tax Reform Act of 1986[86] eliminated most of the benefits of capital gains to shareholders, and repealed the *General Utilities*[87] rule permitting corporations to distribute appreciated assets in liquidation without incurring a corporate tax. Accordingly, Section 304's historical function is largely obsolete, and the most intriguing aspect of the formation of strange loops is their potential for enabling a corporation to dispose of a partial interest in an affiliate without incurring a corporate tax.

Much of the analysis here is premised on the idea that indirectly self-owned stock is a kind of quasi-treasury stock that properly ought to be regarded as not outstanding for tax purposes. Others have voiced this idea, especially in connection with the zero basis problem

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[86] Pub. L. No. 99-514, 100 Stat. 2085.

[87] General Util. & Operating Co. v. Helvering, 296 U.S. 200 (1935).

discussed in Part III above.[88] Past commentary has focused on a wholly-owned subsidiary dealing in the shares of its parent. Any workable rule, however, has to deal with the possibility of minority interests and more intricate cross ownership arrangements. Indeed, the more celebrated recent instances of tax-oriented transactions involving strange loops, those involving May Department Stores and McDermott, Inc., made use of minority interests. In these contexts, if indirectly owned stock is to be treated as not outstanding, significant changes need to be made in both the circumstances in which corporations recognize gain or loss upon realignments of ownership interests, and in the manner in which any gain or loss is calculated.

The four sections below discuss four ways that a strange loop can arise. In each section, the discussion addresses the relatively simple situation involving only two corporations. (The general case is far more complicated.) The goal is to understand better how indirect self ownership affects the transaction, and to speculate how the tax law might behave if it were to give due recognition to the indirect self ownership that strange loops represent. No attention is given here to the treatment of the individual shareholders; the focus instead is on the corporations.

A.   *Issuer Sells its Shares to an Affiliate for Cash*

Suppose C_1 owns 30% of C_2, and C_2 pays \$60 to C_1 for newly issued shares representing a 40% interest, as shown in Figure 14.

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[88]   *See* ALI, FEDERAL INCOME TAX PROJECT, SUBCHAPTER C, PROPOSALS ON CORPORATE ACQUISITIONS AND DISPOSITIONS 302–04 (1982); ABA Tax Sec, Committees on Sales, Exchanges and Basis and Corporate Stockholder Relationships, *Tax Sec. Recommendation No. 1980-8*, 33 TAX LAW. 1543 (1980); N.Y. ST. BA. ASS'N TAX SEC., *Sale or Exchange by a Subsidiary Corporation of Its Parent Corporation's Stock,* 47 TAXES 146 (1969).

FIGURE 14

before   after
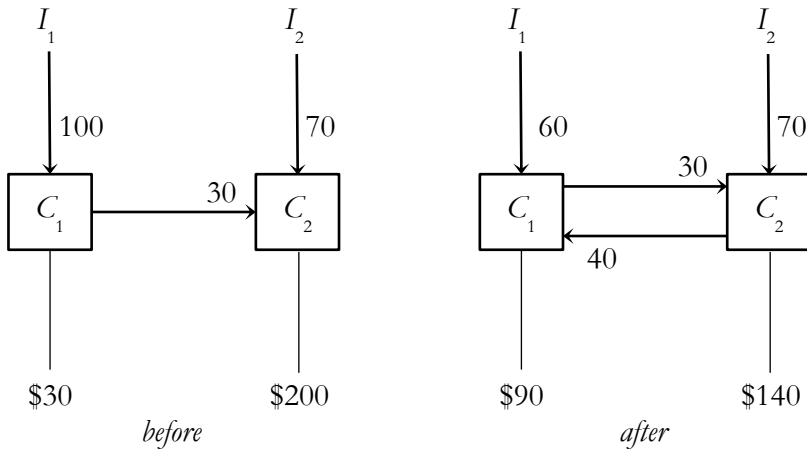
To some extent, C_2 has acquired a nonasset: 30% of the stock of C_1 that it has acquired represents a 12% indirect self-ownership interest. Only 88% of C_2's shares remain outstanding if the 12% self-owned interest is disregarded. So the transaction plausibly might be viewed as a redemption by C_2 of some of its shares owned by C_1. Of the 88% non-self-owned interest in C_2, C_1 holds 20.45% ((30-12)/88), and I_2 holds the other 79.55% (70/88). Part of the cash received by C_1 could be considered to be proceeds of a redemption to the extent of the newly self-owned 12% interest.

To see this in formula terms, let $Z(C_i)$ represent the SELF-OWNERSHIP interest of C_i. In the two corporation case,

(38) $$Z(C_1) = Z(C_2) = S(C_1,C_2)S(C_2,C_1).$$

Because the degree of self-ownership is the same for both corporations, it can be referred to simply as Z in the two corporation case.[89] When the strange loop is formed, C_1's interest in C_2 drops from $S(C_1,C_2)$ to $(S(C_1,C_2) - Z)/(1 - Z)$.

In the example, the $60 cash received upon the issuance of the C_1 stock to C_2 would be treated as the proceeds of a redemption to the

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[89]   In the general case, $Z(C_i) = 1 - (1/F(C_i,C_i))$.

extent that C_1's interest in C_2's underlying asset value declines from $60 (30% of $200) to $28.63 (20.45% of $140), a decline of $31.37. The balance properly would be covered by Section 1032. C_1's continuing 20.45% interest is only 68.17% of its 30% initial interest, so the redemption would be treated as a sale under Section 302.[90] The $31.37 sold is 52.3% of C_1's initial interest of $60, so 52.3% of C_1's basis would be deducted from the redemption proceeds.

The implicit disposition by C_1 of part of its interest in C_2 is far greater if I_2's interest in C_2 was acquired with the funds used by C_2 to purchase the C_1 stock and as part of a single plan. Ordinarily, I_2's purchase of newly issued C_2 stock has no effect on C_1, even though C_1's interest is diluted, since there has been a genuine pooling of investment, and C_1 does not realize anything on the transaction. Neither of these is true, however, if the funds raised on the stock issuance are used to purchase stock of C_1: There has been no pooling of investment from C_1's point of view, since the only asset acquired by C_2 is stock of C_1, which is a nonasset to C_1; C_1 has realized something on the transaction by its receipt of cash from C_2. In such a case, it would be appropriate to measure the extent of C_1's disposition of its interest in C_2 by comparing its interest in C_2's underlying assets before the issuance of shares to I_2 with its interest in these assets after the transactions are fully completed.

This reasoning underlies the "deemed redemption" rule of Notice 89-37,[91] but with a different kind of entity: In the Notice, C_2 is a partnership. Nothing in the formal theory requires the members of $\{C\}$ to be corporate entities, and its theorems and formulae apply equally well to "straight up" partnerships with a single class of inter-

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[90] I.R.C. § 302(b)(2) provides that a redemption is treated as a sale, rather than a dividend, if the redeeming shareholder's interest declines by more than 20% as a result of the redemption, and the shareholder owns less than a 50% interest (by voting power) after the redemption.

[91] 1989-1 C.B. 679. For a critique of the Notice, which is generally supportive of the deemed redemption rule, see N.Y. St. Ba. Ass'n Tax Sec., *Report on Notice 89-37*, 46 Tax Notes 99 (Jan. 1, 1990).

ests, or to mixed sets of corporations and partnerships. Moreover, forming a strange loop with partnerships has its attractions as a tax matter, since partnerships do not add a layer of intercorporate taxation, and are potentially easier to unwind tax-free. When C_2 is viewed as a partnership, the transaction described in Figure 14 resembles the May Department Stores transaction, which prompted the issuance of Notice 89-37. In that transaction, May's initial interest in the partnership's assets was 100%, since all of the funds used by the partnership to acquire an interest in May were supplied by the other partner as part of the overall transaction.

The aggregate theory of partnerships makes it particularly easy to view stock of a partner held by a partnership as partially self-owned. Corporations more often are viewed as separate entities rather than as aggregates, although an aggregate theory of corporations underlies stock attribution rules such as those used in applying Section 382, and the aggregate theory plays an important role in the consolidated return regulations.

As promised in the Notice, the Service has issued proposed regulations that would treat the transaction described in Figure 14 as a sale by C_1 to the extent of its non-retained interest in C_2, where C_2 is a partnership.[92] Under the proposed regulations, it appears that the Service would measure C_1's retained interest in C_2 for this purpose by the full 30% direct ownership percentage, even though it represents only 20.45% of the non-self-owned interest in C_2. The proposed regulations therefore properly identify the 70% dilution that occurs upon the issuance of the C_2 shares as part of the plan, but miss the additional 9.55% dilution in C_1's interest that occurs by reason of the strange loop.

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[92] Prop. Treas. Reg. § 1.337(d)-3, 57 Fed. Reg. 59,324 (Dec. 15, 1992). These regulations are issued under the authority of I.R.C. Section 337(d), which authorized the Service to issue regulations that may be necessary or appropriate to carry out the purposes of the repeal of the *General Utilities* doctrine.

Moreover, the proposed regulations characterize the deemed sale as a "redemption" of C_1's stock, although the C_1 stock considered to be redeemed is itself issued as part of the transaction. Perhaps it would be more accurate to say that a portion of the C_1 stock that was purported to have been issued was in fact not issued because it remained indirectly self-owned, and that the cash received, to the extent attributable to the not-really-issued stock, is characterized more properly as the proceeds of a sale of part of C_1's interest in C_2.

In Figure 14, there is a deemed redemption as well as a deemed sale, but it is C_2's stock and not C_1's, that is being redeemed implicitly upon the purchase of C_1 stock. It is the deemed redemption of C_2's stock that accounts for the difference between the 30% direct interest and the 20.45% interest in the non-self-owned stock. The deemed redemption of C_2's stock is relatively inconsequential if C_2 has only a small interest in C_1, which ordinarily would be the case where C_1 is a large public corporation and C_2 is a relatively small joint venture. When the two corporations are more equal in size, however, the deemed redemption of C_2 is more pronounced.

To the extent that C_2 has purchased a self-owned interest, the self-owned interest should be regarded as a non-asset (like Treasury stock) and should therefore have no basis. (This is very different from treating it as an asset with zero basis.) C_2's basis in the C_1 stock should be limited to $A(C_1)(S(C_2,C_1) - Z)/(1 - Z)$, which is C_2's share of C_1's underlying assets. This basis becomes relevant if C_2 later unwinds the strange loop, or if additional cross ownership develops that has the effect of a deemed disposition.

B.   *Issuer's Shareholders Sell Shares to an Affiliate*

Suppose that in the example illustrated in Figure 14, C_2 had acquired its interest in C_1 from C_1's shareholder I_1 rather than from C_1 directly. With this variation, C_1 itself realizes nothing from the transaction, since the cash goes to its shareholders, but the amount of its

effectively outstanding (that is, not indirectly self-owned) stock goes down to 88% of its previous level. In addition, the portion of C_2's outstanding (in a similar sense) stock that C_1 holds for the benefit of its own shareholders goes down from 30% to 20.45%. This last figure can be seen as the product of I_1's percentage interest in C_1 (60%) times C_1's flow from C_2 (0.30/0.88, or 0.3409). This figure coincides with the amount computed in the preceding section as C_1's retained interest in C_2. Where the cash used for the purchase is "old and cold" cash of C_2, the decrease in C_1's interest in C_2 is limited to this amount.

If instead I_2 acquired the 70% interest in C_2 by contributing the cash used to purchase the C_1 stock as part of a single plan, then C_1's overall dilution of its interest in C_2 should be taken into account. Since before I_2 acquired the 70% interest in C_2, C_2 was wholly owned by C_1, the transactions together have the effect of decreasing C_1's interest in C_2 from 100% to 20.45%. This is the same result that occurs when C_2 purchases stock directly from C_1, as described in the preceding Part, if both the deemed sale by C_1 and the deemed redemption of C_2 stock are taken into account.

One way to view the proper tax treatment of C_1 is to recast the transaction so that C_1 is considered to have distributed 70% of C_2 to I_1, which I_1 is considered to have sold back to C_2 along with 40% of the stock of C_1. C_1 then would recognize gain, if any, on the distributed interest, as well as on the small further amount (12% in the example) that is deemed to be a redemption of C_2 stock by C_1.

Figure 15 shows a sequence of transactions that illustrates the interplay between value, basis and self-ownership as a strange loop is first formed and later strengthened.

FIGURE 15

(A) initial position

(B) purchase of a 50% interest

(C) purchase of another 10% interest
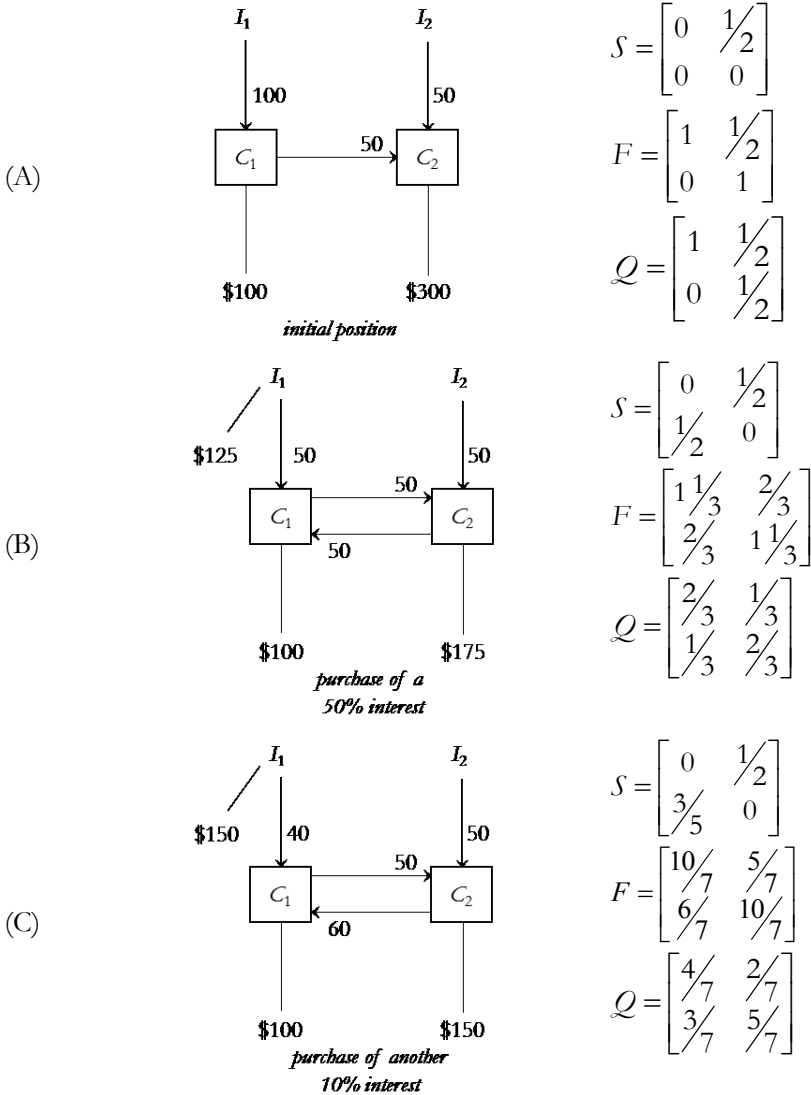
In Figure 15B, C_2 used \$125 of its funds to acquire a 50% interest from I_1. After the acquisition, C_2 indirectly owns 25% of itself, so C_1's direct shareholding represents one-third of the non-self-owned inter-

est $((50 - 25)/(100 - 25))$, and I_2's shareholding represents the remaining two-thirds. (These values are shown in Figure 15 in matrix Q, which is explained more fully in Part VII.C below.) In Figure 15A there is no self-ownership, and C_1 owns one-half of C_2. Consequently, the purchase from I_1 caused C_1's interest in C_2 to drop from one-half of C_2's initial underlying asset value of $300 to one-third of C_2's remaining non-self-owned value of $225. The initial interest has a value of $150, and the remaining interest has a value of $75, so the difference of $75 represents the value of the interest that C_1 is deemed to have disposed of. This value represents one-half of the value of C_1's initial interest, so one-half of C_1's basis in C_2 can be applied against this deemed amount realized. If C_1's initial basis was $100 (so that the initial value of $150 includes appreciation of $50), then the recognized gain is $75 minus $50, or $25. C_1's basis after the transaction is $50 ($100-$50). Although C_2 has paid $125 for the purchased C_1 stock, its basis is limited to $50, which is the portion of the stock that is attributable to C_2's one-half direct interest in C_1's underlying assets of $100. The balance of C_2's payment is $75, which represents the amount paid for the implicit redemption of 25% of C_2's total value of $300.

In Figure 15C, C_2 acquires another 10% direct interest from I_1 for $25. This additional purchase causes C_1's interest in the non-self-owned value of C_2 to decline from one-third of $225, or $75, to two-sevenths of $210, or $60. The $60 value of the retained interest is 80% of the $75 value of C_1's interest before this second transaction, so 20% of C_1's basis of $50, or $10, can be deducted from a deemed amount realized of $15 ($75 minus $60) to yield a gain of $5.

C. *Issuer and an Affiliate Exchange Shares*

Any two corporations can form a strange loop by issuing shares to each other. Figure 16 is a variant of Figure 14, where the two corporations have no prior relationship.

FIGURE 16

before		*after*
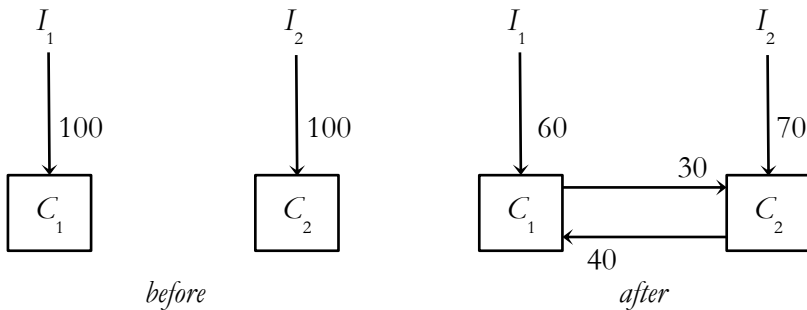
Each corporation is, from its own point of view, doing two things: It is issuing indirectly self-owned stock, and it is issuing additional stock in exchange for an interest in the other corporation's underlying assets. It is hard to make a case that any of this should be taxable. The issuance of indirectly self-owned stock should have no tax consequences, and the issuance of other stock should be tax-free under Section 1032.

Each corporation should have some basis in its stock of the other, but only to the extent that the stock represents an interest in the other corporation's underlying assets rather than a self-owned interest. Before considering how this basis should be calculated, it is necessary first to determine how the cross ownership percentages relate to the relative values of the underlying assets. At first, it might seem that the cross ownership percentages should be proportional to the values of the underlying assets. If, as in Figure 16, C_1's interest in C_2 is three-fourths of C_2's interest in C_1, then one might expect C_1's underlying assets to be worth three-fourths as much as C_2's. This relationship is approximately valid for very small cross ownership percentages, but does not hold, even approximately, in cases of more extensive cross ownership, such as the example in Figure 16.

To determine the correct relationship, we must return to the original valuation equations for C_1 and C_2:

(39)
$$V(C_1) = A(C_1) + S(C_1,C_2)V(C_2);$$
$$V(C_2) = A(C_2) + S(C_2,C_1)V(C_1).$$

The right half of the second equation in Equation (39) can be substituted for $V(C_2)$ in the first:

(40) $V(C_1) = A(C_1) + S(C_1,C_2)\big(A(C_2) + S(C_2,C_1)V(C_1)\big).$

Solving Equation (40) for $V(C_1)$ yields:

(41) $$V(C_1) = \frac{A(C_1) + S(C_1,C_2)A(C_2)}{1 - S(C_1,C_2)S(C_2,C_1)}$$

There is another constraint on the value of $V(C_1)$, since the value of I_1's interest in C_1 before and after the exchange must be the same. Before the exchange, this value is equal to C_1's underlying asset value $A(C_1)$, and after the exchange, it is equal to I_1's direct interest in the total value of C_1:

(42) $A(C_1) = M(I_1,C_1)V(C_1) = \big(1 - S(C_2,C_1)\big)V(C_1).$

Solving Equation (42) for $V(C_1)$ yields:

(43) $$V(C_1) = \frac{A(C_1)}{1 - S(C_2,C_1)}.$$

Since the right halves of Equations (41) and (43) are both equal to $V(C_1)$, they are equal to each other:

(44) $$\frac{A(C_1)}{1 - S(C_2,C_1)} = \frac{A(C_1) + S(C_1,C_2)A(C_2)}{1 - S(C_1,C_2)S(C_2,C_1)}.$$

Equation (44) can be solved for the ratio of $A(C_1)$ to $A(C_2)$:

(45) $$\frac{A(C_1)}{A(C_2)} = \frac{S(C_1,C_2)\big(1 - S(C_2,C_1)\big)}{S(C_2,C_1)\big(1 - S(C_1,C_2)\big)}.$$

To make this formula more intuitive, we need a new concept $Q(C_i,C_j)$, the OUTSIDE INTEREST, measures the interest in C_j that C_i holds on behalf of its individual shareholders:

(46) $$Q(C_i,C_j) = F(C_i,C_j)\sum_k M(I_k,C_i).$$

The outside interest is the percentage of the flow from C_j to C_i that is distributable to C_i's individual shareholders. It is fairly straightforward to show the following:

Theorem 4: The outside interests of all corporations in any given corporation add up to 100%:

$$\sum_i Q(C_i, C_j) = 1, \text{ for all } j.$$

Appendix I contains a proof.

In the two-corporation case, all of the stock not owned by individuals is owned by the other corporation. Hence,

(47)
$$S(C_1, C_2) = 1 - \sum_k M(I_k, C_2);$$
$$S(C_2, C_1) = 1 - \sum_k M(I_k, C_1).$$

The general flow matrix F for two corporations can be stated as follows:

(48)
$$F = \begin{bmatrix} \dfrac{1}{1 - S(C_1,C_2)S(C_2,C_1)} & \dfrac{S(C_1,C_2)}{1 - S(C_1,C_2)S(C_2,C_1)} \\[4mm] \dfrac{S(C_2,C_1)}{1 - S(C_1,C_2)S(C_2,C_1)} & \dfrac{1}{1 - S(C_1,C_2)S(C_2,C_1)} \end{bmatrix}$$

Using this general flow matrix, it is possible to construct the general outside interest matrix:

(49)
$$Q = \begin{bmatrix} \dfrac{1 - S(C_1,C_2)}{1 - S(C_1,C_2)S(C_2,C_1)} & \dfrac{S(C_1,C_2)(1 - (C_1,C_2))}{1 - S(C_1,C_2)S(C_2,C_1)} \\[4mm] \dfrac{S(C_1,C_2)(1 - (C_1,C_2))}{1 - S(C_1,C_2)S(C_2,C_1)} & \dfrac{1 - S(C_1,C_2)}{1 - S(C_1,C_2)S(C_2,C_1)} \end{bmatrix}$$

From the matrix and Equation (45), it becomes apparent that:

(50)
$$\frac{A(C_1)}{A(C_2)} = \frac{Q(C_1,C_2)}{Q(C_2,C_1)}.$$

Equation (50) makes clear that what is proportional to the ratio of underlying assets is not the direct cross ownership percentage interests, but the cross ownership interests based on outside interests. Stated more intuitively, after the exchange C_1's outside interest in C_2's

underlying assets must be equal to C_2's outside interest in C_1's underlying assets.

These equations can be used to determine the value of the exchanged interests, and the amount of basis each corporation should have for its stock of the other corporation. Assume for example, that C_1's underlying assets are worth \$90, and C_2's underlying assets are worth \$140 (the ratio of asset values assumed here is consistent with the ratio of outside interests). From Equation (43), $V(C_1)$ after the exchange is \$90/(1 – 0.6), which is \$150. Accordingly, the 40% interest issued to C_2 has a value of \$60. Similarly, $V(C_2)$ after the exchange is \$140/(1 – 0.7), which is \$200, and the 30% interest issued to C_1 also has a value of \$60. To determine basis, the outside interests must first be computed:

$$Q(C_1,C_2) = \frac{S(C_1,C_2)(1-S(C_2,C_1))}{1-S(C_1,C_2)S(C_2,C_1)} = \frac{0.3(1-0.4)}{1-0.3\times0.4} \approx 20.45\%;$$

(51)

$$Q(C_2,C_1) = \frac{S(C_2,C_1)(1-S(C_1,C_2))}{1-S(C_1,C_2)S(C_2,C_1)} = \frac{0.4(1-0.3)}{1-0.3\times0.4} \approx 31.82\%.$$

C_1's interest in C_2's underlying assets is therefore 20.45% of \$140, or \$28.63. C_2's interest in C_1's underlying assets is 31.82% of \$90, which is also \$28.63. These amounts represent the proper basis of each corporation in the other's shares when the self-ownership interests are excluded.

D. *Issuer's Shareholders Exchange Shares for an Affiliate's Shares*

Where C_2 issues its own stock to C_1's shareholders in exchange for C_1 stock, the effect is much the same as the cash purchase discussed in Part VII.B, where the cash itself was the proceeds of an issuance of C_2 stock. The only difference in the case of the exchange is that the person acquiring the C_2 stock is the same as the person surrendering

the C_1 stock. While this difference might have some bearing on the treatment of the individual shareholders involved,[93] it has no effect on the relationships between the two corporations. Consequently, C_1 here has made a deemed distribution of a portion of its interest in C_2, which can be measured by the difference in C_1's outside interest in C_2 before and after the transaction.

FIGURE 17

Exchange of shares of McDermott, Inc. ("M")
for shares of McDermott International, Inc. ("I")

before   *after*
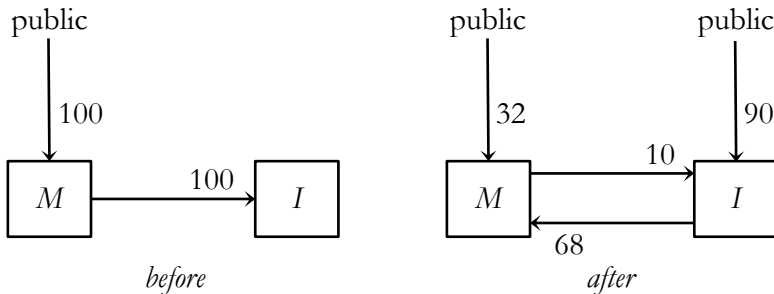
Matrices after the exchange

$$S = \begin{bmatrix} 0 & 0.10 \\ 0.68 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 1.073 & 0.107 \\ 0.730 & 1.073 \end{bmatrix} \quad Q = \begin{bmatrix} 0.343 & 0.034 \\ 0.657 & 0.966 \end{bmatrix}$$

The best known example of a strange loop being formed in this manner is the 1982 exchange offer by McDermott International, Inc. ("International") for shares of its parent, McDermott, Inc. ("McDermott"). In that exchange, International issued shares representing a

[93] The McDermott transaction, discussed *infra* at text accompanying notes 94–97, gave rise to litigation regarding the proper treatment of the exchanging shareholders. Bhada v. Comm'r, 89 T.C. 959 (1987), *aff'd*, 892 F.2d 39 (6th Cir. 1989). The issue in *Bhada* was whether Section 304 applied to the transaction even though the only consideration delivered to the exchanging shareholders was stock of the acquiring corporation. Both courts involved determined that Section 304 did not apply, but the decisions were based more on a technical reading of the relevant statutes than on any deemed distributions that might be imputed upon the formation of a strange loop.

90% ownership interest in itself, in exchange for 68% of the shares of McDermott.[94] (The small amount of cash that also was delivered in the exchange is ignored here.) International was wholly owned by McDermott before the exchange; McDermott retained its pre-existing interest in International, but that interest was diluted to 10%. The diagram and matrices are shown in Figure 17.

Glancing at the diagrams alone, it might appear that the formation of the strange loop has caused McDermott to divest itself of 90% of its interest in International. Yet, when the indirectly self-owned stock is disregarded, McDermott's implicit divestiture is even greater. Out of the retained 10% interest in International, 68% is attributable to indirectly self-owned stock, and only 32% is attributable to International's other assets. Thus, only 10% of 32%, or 3.2% represents a true continuing interest in International. This percentage needs to be adjusted, however, because 6.8% of both McDermott and International is indirectly self-owned after the exchange. Accordingly, the 3.2% figure should be grossed up by 93.2% (100% minus 6.8%), which raises it to 3.4%. This final figure can be read directly off matrix Q in Figure 17, which shows 3.4% to be the continuing outside interest of McDermott in International.

The McDermott transaction prompted the enactment of Section 1248(i), to prevent corporations from using the technique to distribute interests in controlled foreign corporations with accumulated E&P that had not yet been subject to U.S. tax.[95] Section 1248(i) applies to an exchange by C_1's shareholders of their C_1 stock for newly issued C_2 stock, if C_2 is a controlled foreign corporation and C_1 owns at least 10% of its stock. Section 1248(i) recharacterizes such an exchange, for purposes of Section 1248, as if the C_2 stock issued in the exchange had first been issued to C_1 and then distributed by C_1 to its shareholders.

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[94]   Bhada v. Comm'r, 89 T.C. 959, 962 (1987).

[95]   Pub. L. No. 98-369, § 133(a), 98 Stat. 494, 667–68 (1984).

The difficulty with this formulation is that the value of the C_2 stock issued in the exchange may well exceed the value of C_1's previously held interest in C_2, since the C_2 stock issued in the exchange represents an indirect ownership interest in C_1's other assets as well as a direct ownership interest in C_2. The legislative history interprets Section 1248(i) in a way that limits the amount of the deemed distribution to the value of C_1's previously held interest in C_2, but it does so by means of an example that leaves some questions unanswered.[96] The example is shown in Figure 18.

FIGURE 18

before   *after*
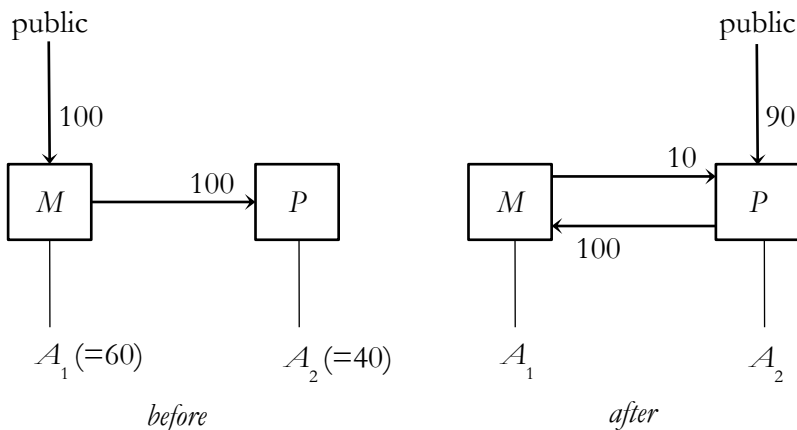
Matrices after the exchange

$$S = \begin{bmatrix} 0 & 0.10 \\ 1 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 1.111 & 0.111 \\ 0.111 & 1.111 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

In this example, unlike the McDermott exchange, the acquiror P acquires 100% of M, rather than the 68% that was acquired by Inter-

96   JOINT COMM. ON TAX'N, GENERAL EXPLANATION OF THE REVENUE PROVISIONS OF THE DEFICIT REDUCTION ACT OF 1984, 98th Cong., 2d Sess. 448 (Comm. Print 1984) [hereinafter 1984 BLUEBOOK].

national.[97] Consequently M effectively has distributed its entire previ-ously held interest in P, and its retained 10% interest is nothing more than stock that is indirectly self-owned by P. M's outside interest in P after the exchange, as shown by matrix Q, is zero, which is hardly sur-prising, since M no longer has any outside shareholders.

The 1984 Bluebook, after presenting this example, states:

> The effect of the Act is to treat the excess of the value held by the former M shareholders after the exchange ($100 mil-lion) over the amount by which P's value was augmented ($60 million) as if M had distributed P shares equal to that differ-ence ($40 million in the example) to its shareholders.[98]

While the Bluebook comes to the right conclusion, its terminology is slightly misleading. P's value $V(P)$ was augmented in the exchange by $71 million, not $60 million, but the extra $11 million is attributable to the self-owned interest.

The first question left unanswered by the example is whether the deemed distribution implied by Section 1248(i) causes the parent to recognize gain under Section 311(b) in excess of its share of the sub-sidiary's accumulated E&P. The statutory language suggests otherwise, since Section 1248(i) applies only for purposes of Section 1248.[99] The example avoids the issue, however, because all of the appreciation in the subsidiary's stock is assumed to be attributable to its accumulated E&P.

More significantly, the example does not address how much is deemed to have been distributed if, as in the McDermott transaction,

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[97] International acquired the remaining shares of McDermott in a subsequent merger. See Amendment No. 1 to Form S-l, Registration Statement Under the Securities Act of 1933, McDermott Incorporated and McDermott International, Inc. (Feb. 14, 1983).

[98] 1984 BLUEBOOK, *supra* note 96, at 448.

[99] I.R.C. § 1248(i)(1). Pending legislation would clarify this point. See Technical Tax Simplification and Corrections Act of 1993, H.R. 3419, 103d Cong., 1st Sess. § 1001(g)(1)(D).

less than 100% of the former parent's shares are acquired by its subsidiary. A partial acquisition is considered in Notice 89-37,[100] where the acquiring entity, a partnership, acquires stock of a partner in exchange for a 70% interest in itself. The Notice assumes that the exchange causes the partner to dispose of 70% of its interest in the partnership, but the deemed disposition is actually greater, because a portion of its 30% retained interest is allocable to the stock held by the partnership. If, as in Figure 14, the stock acquired by the partnership represents a 40% interest in the partner, then the corporate diagram and flow matrices would be as shown in Figure 19.

The matrix Q confirms the conclusion reached in Part VII.A above in connection with a cash purchase: C's outside interest in P after the transaction is only 20.5%, rather than 30%, as the Notice concludes. If indirectly self-owned stock is to be truly ignored, then looking only at the direct retained interest understates the real extent of the implicit divestiture caused by the formation of the strange loop.

FIGURE 19

before   *after*
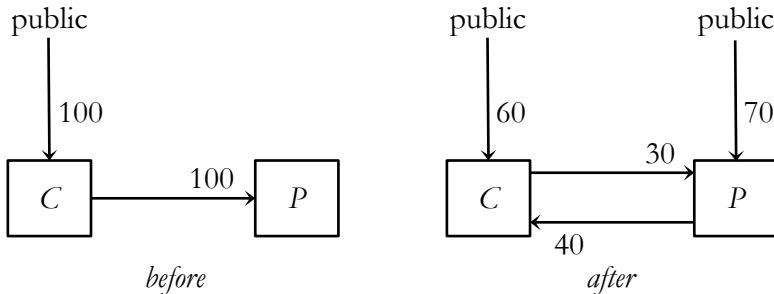
Matrices after the exchange

$$S = \begin{bmatrix} 0 & 0.30 \\ 0.40 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 1.136 & 0.341 \\ 0.454 & 1.136 \end{bmatrix} \quad Q = \begin{bmatrix} 0.682 & 0.205 \\ 0.318 & 0.795 \end{bmatrix}$$

[100] 1989-1 C.B. 679.

The Service has proposed to attack McDermott-type transactions at the shareholder level. In Notice 94-46,[101] the Service announced its intention to issue regulations under Section 367 that would require U.S. shareholders to recognize gain upon an exchange of shares of a domestic corporation for shares of its foreign subsidiary. Such an exchange can cause the former subsidiary to cease to be a controlled foreign corporation, causing the former parent to be no longer subject to current tax on the subsidiary's subpart F income. This approach is a dubious use of Section 367: The real issue is the implicit divestiture by the former parent of an interest in its former subsidiary. While it might be argued that this divestiture should trigger a corporate-level tax (such as that provided by Section 1248(i)), the amounts of the shareholders' gains are irrelevant to this issue.

More recently, the Service has addressed the former parent's implicit divestiture of its interest in the subsidiary, but only to a limited degree. In a series of private letter rulings, the Service had previously approved corporate "inversions" in which the former parent's interest in the former subsidiary was significantly diluted by the subsidiary's issuance of shares to the former parent's shareholders.[102] If the shareholders of the former parent exchange all of their shares in the transaction, as in Figure 18, any dilution has no effect on the shareholders of the former subsidiary, who now own, directly or indirectly, all of both corporations. Yet, the dilution causes the strange loop, and the attendant tax cost of cycling dividends, to dwindle into insignificance. For example, in Figure 18, P could have issued 10 times as many shares in the transaction, causing M's retained interest to be 1% rather than 10%.

The Service has announced that it will issue guidance that treats dilution of the former parent's interest as a divestiture to the extent

[101] 1994-18 I.R.B. 7.

[102] I.R.S. Priv. Ltr. Ruls. 1992-38-009 (Mar. 13, 1992), 1990-22-038 (Mar. 5, 1990), 1988-27-061 (Apr. 13, 1988).

that it reduces the value of the interest.[103] This result is perfectly appropriate, because the former parent's acceptance of dilution that reduces the value of its interest in the former subsidiary is in effect a distribution of the diminished value by the former parent to the former subsidiary. Yet, this guidance does not address the implicit divestiture that occurs even in the absence of this dilution. For example, in Figure 18, even if *M*'s retained interest in *P* has the same value as its interest in *P* before the exchange, that interest has been converted into a mere link in *P*'s indirect self-ownership interest.

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[103] Notice 94-93, 1994-2 C.B. 563.

VIII. Unwinding a Strange Loop

When a corporation sells stock that represents part of a strange loop, it is selling, in part, an interest in itself. The unwinding of the strange loop, by decreasing the corporation's indirectly self-owned stock, effectively increases the amount of stock outstanding. Section 1032 plausibly could be extended to provide for no gain or loss on the sale of such stock, to the extent it represents the sale of a previously self-owned interest. This Part parallels the preceding Part VII and considers four possible ways that a strange loop can disappear.

A. *Issuer Repurchases its Stock From an Affiliate*

In Figure 14, imagine running the film backward: The parties move from the "after" side to the "before" side of Figure 14 as a result of C_1's redemption for cash of the C_1 stock held by C_2. Ignoring the strange loop, the redemption would have no tax consequences to C_1, and C_2 would recognize gain or loss equal to the difference between the redemption price and its basis in the C_1 stock redeemed.

When the strange loop is taken into account, however, the transaction takes on a different cast. In addition to redeeming its own stock, C_1 is purchasing an additional interest in C_2: As seen in Section VIII.A, C_1's interest in C_2's underlying assets increases by \$31.37, from 20.45% of \$140 to 30% of \$200. Since this portion of the \$60 cash payment has the effect of increasing C_1's interest in C_2, presumably \$31.37 should be added to C_1's basis in the C_2 stock.

The \$28.63 balance of the cash payment would be treated as an amount distributed in redemption of C_1's stock. From C_2's point of view, this is the only part of the cash received that should be treated as proceeds of the redemption, and the balance should be regarded as the tax-free issuance of the previously indirectly self-owned stock of C_2. This limitation on the amount of the cash payment that is treated

as proceeds of the redemption by C_1 is consistent with the determination of C_2's basis in the C_1 stock that it acquired upon the formation of the strange loop, which was limited to the value of $Q(C_2,C_1)A(C_1)$ at that time. Accordingly, the only gain or loss recognized by C_2 upon the redemption is C_2's share (based on its outside interest) of any appreciation or depreciation in C_1's underlying assets during the period that C_2 held the C_1 stock.

B.   *Affiliate Sells an Issuer's Shares to a Third Party*

A strange loop between two corporations can be unwound, in whole or in part, if one of the corporations sells stock in the other corporation to a third party. An example of a two stage unwind can be seen by reversing the transactions in Figure 15. Thus, starting with the position in Figure 15C, C_2 sells a 10% interest in C_1 to I_1 for $25. This sale causes the value of C_1's outside interest in C_2 to increase from $60 (two-sevenths of $210) to $75. Since the cash is coming from C_1's shareholders, this increase in interest can be regarded as arising from a $15 deemed contribution to C_1 from I_1 which would be added to C_1's basis in its shares of C_2. At the same time, the sale causes the value of C_2's indirectly self-owned interest to decline from $90 (30% of $300) to $75 (25% of $300), so $15 of the cash received can be regarded as the sale of "quasi-treasury stock" that would be tax-free by analogy to Section 1032, and the $10 balance would be the proceeds of the disposition of 10% of C_2's interest in C_1's underlying asset value of $100.

Similarly, if C_2 sells an additional 50% direct stock interest in C_1 for $125, thereby shifting from Figure 15B to Figure 15A, C_1's outside interest increases from $75 (one-third of $225) to $150 (one-half of $300). This $75 would be deemed to be contributed from I_1 so C_1's basis in its shares of C_2 would increase by that amount. If these transactions occurred after the sequence initially described in Figure 15, C_1's basis in its shares of C_2 would be $40 before C_2's sale of the 10%

interest described in the preceding paragraph, and would increase to $55 upon that sale, and to $130 upon the sale of the additional 50% interest. This $130 basis is $30 more that the basis that C_1 had before the sequence of transactions initially described in Figure 15, but C_1 recognized $30 in gain as a result of those transactions. The sale of the additional 50% interest eliminates C_2's self-owned interest of $75, and therefore $75 of the $125 received on the sale should be excluded from income under an extension of Section 1032. The balance of $50 represents the disposition of C_2's 50% interest in C_1's underlying asset value of $100.

C. *Issuer and an Affiliate Redeem Each Other's Shares*

When two corporations in a strange loop redeem each other's shares, they are partly exchanging interests in each other's underlying assets, and partly cancelling self-owned interests. The former should be a taxable disposition, and the latter, a tax nonevent. In the past, the Service has treated such an exchange as wholly tax-free, but in view of the repeal of the *General Utilities* doctrine and the Service's approach to May-type transactions, as set forth in Notice 89-37,[104] it is likely that the Service today would regard these transactions as wholly taxable.

Figure 20 shows the transaction described in Revenue Ruling 79-314,[105] in which two corporations owning stock in each other exchanged shares, terminating the cross ownership.

The Service observed that each corporation was participating in the exchange both in its capacity as a redeeming shareholder and in its capacity as an issuer transferring appreciated property in redemption of its own stock. The exchange would have been taxable to both corporations in their capacity as redeeming shareholders, but in their capacity as issuers redeeming their own stock, the transaction would

[104] 1989-1 C.B. 679.

[105] 1979.2 C.B. 132. The transaction described in the ruling also contained a cash payment to equalize share values, which is omitted from the discussion here.

have been tax-free under Section 311(a), as then in effect. The Service ruled that the nonrecognition rule of Section 311(a) took precedence here, so no gain or loss was recognized by either corporation. A similar conclusion was reached in Revenue Ruling 80-101,[106] where the cross ownership was terminated in connection with a liquidation of one of the corporations.

FIGURE 20

$$
I_1 \qquad\qquad I_2 \qquad\qquad I_1 \qquad\qquad I_2
$$

88   86   100   100

14

C_1 C_2 C_1 C_2

12

before   *after*
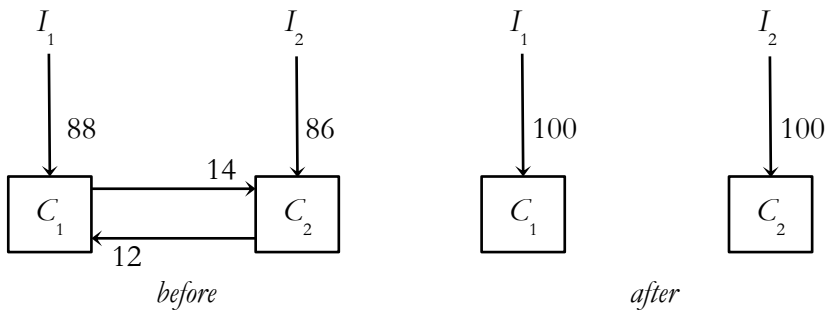
Matrices before the exchange

$$
S = \begin{bmatrix} 0 & 0.14 \\ 0.12 & 0 \end{bmatrix} \quad
F = \begin{bmatrix} 1.017 & 0.142 \\ 0.122 & 1.017 \end{bmatrix} \quad
Q = \begin{bmatrix} 0 & 0.125 \\ 0.105 & 0 \end{bmatrix}
$$

It is doubtful that the Service would reach the same conclusions today, since the repeal of the *General Utilities* doctrine causes distributions of appreciated property in redemption or liquidation of the issuer's stock to be taxable to the issuer.[107] Indeed, the Service's views on this matter can be seen more directly in its treatment in Notice 89-37 of the "back end" of the May Department Stores transaction. That transaction, in which a partnership with May as one of its two partners acquired stock of May, has been widely assumed to be followed eventually by a second transaction in which the partnership will distribute the May stock back to May in redemption of its partnership

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[106] 1980-1 C.B. 70.

[107] I.R.C. § 311(a).

interest.[108] In that second transaction, May would be acting both in its capacity as a partner receiving property in redemption of its partnership interest (which would be tax-free under Section 731 (a)) and as an issuer transferring an appreciated partnership interest in redemption of its own stock (which would be taxable under Section 311 as now in effect). In contrast to the two rulings discussed in the preceding paragraph, the Service announced in Notice 89-37 that it would issue regulations providing that in such a case the recognition rule would take precedence (the "deemed distribution rule").

The Service's propounding of the deemed distribution rule is understandable in view of its mandate, expressed in Section 337(d), to issue regulations ensuring that the purposes of the *General Utilities* repeal are not circumvented by the application of other provisions of law. Yet, the deemed distribution rule is overbroad to the extent that it seeks to tax appreciation in self-owned interests. In the example set forth in Figure 20, C_1 should be taxed only to the extent that it has surrendered its 12.5% outside interest in C_2's underlying assets, so the taxable amount realized by C_1 on the exchange is $Q(C_1,C_2)\ A(C_2)$.

Similarly, the amount realized by C_2 is $Q(C_2,\ C_1)\ A(C_1)$. Such a cross-redemption would be a fair exchange only if the values for $A(C_1)$ and $A(C_2)$ in inverse proportion to the values for $Q(C_1,C_2)$ and $Q(C_2,C_1)$ so that the amounts realized by C_1 and C_2 are the same. In other cases, a compensating cash payment between the corporations will be needed to reach a fair exchange.

In Figure 19, a corporation disposed of a 79.5% interest in a partnership by forming a strange loop. If the strange loop were unwound by an exchange of ownership interests, as would occur in the "back end" of the May transaction, there would be a disposition of the remaining 20.5% interest. Thus, the proposed treatment of the formation and unwinding of strange loops ensures that, over time, the

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[108] The parties prudently have maintained radio silence on this issue.

disposition of the entire interest in the underlying assets of the partnership is taken into account.

D. *Issuer Redeems its Stock With an Affiliate's Shares*

Suppose an issuer redeems its shares by delivering shares of an affiliate that is part of a strange loop with the issuer. Such a redemption is a reversal of an issuer's offer of its own shares in exchange for shares of another corporation that owns all or part of the issuer. Yet, so far as strange loops are concerned, this transaction is no different from a cash sale of the shares to a third party, with the cash being used to repurchase the issuer's shares. Consequently, the transaction can be analyzed in the same manner as the sale described in Part VIII.B above.

IX. Conclusion

Obviously strange loops are complicated. So, sad to say, must the tax law be if it is to deal with them in a way that properly takes into account indirect self-ownership. It is troubling to think that taxpayers must be able to invert matrices in order to calculate their tax liability. Perhaps such a state of affairs would promote math literacy; taxpayer revolt is a more likely outcome.

Having achieved lower rates and a significant degree of base broadening during the 1980's, tax reformers have set their sights on tax simplification. The irony is that the most notorious sources of tax complexity today are the result of well-intended reform efforts, rather than tax breaks for special interests.[109] Typically, the problem lies in trying to give precise content, through regulatory elaboration, to concepts like nondiscrimination, arbitrage profit, ownership change, passive activity or substantial economic effect.

The regulations that implement these concepts might be described as fractal line drawing in the law. Fractals are shapes with boundaries that display greater detail at greater degrees of resolution, ad infinitum.[110] Anyone who has had to deal with these regulations knows the feeling of infinite regress into greater detail.[111]

One can imagine the same thing happening with the concept of indirect self-ownership. Rules that make sense for extensive cross ownership appear far more dubious when two corporations happen to have small portfolio investments in each other. Some line drawing is

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[109] Sheldon D. Pollack, *Tax Reform: The 1980's in Perspective*, 46 TAX L. REV. 489, 535 (1991).

[110] See generally BENOIT B. MANDELBROT, THE FRACTAL GEOMETRY OF NATURE (updated ed. 1983).

[111] This phenomenon has come to be described as "hyperlexis." See Gordon B. Henderson, *Controlling Hyperlexis—the Most Important "Law and . . . ,"* 43 TAX LAW. 177 (1989); Bayless Manning, *Hyperlexis and the Law of Conservation of Ambiguity: Thoughts on Section 385*, 36 TAX LAW. 9 (1982).

needed to establish a threshold. This in turn raises questions about how to deal with multiple classes of stock, the presence of options and even the extent to which strange loops must be taken into account in determining whether the threshold is met. Resolving these questions easily can lead to further questions about valuation, intent and the like.

An even more fundamental problem is that an attempt to deal coherently with strange loops must coexist with a corporate tax system that is already suffering strains of the post-*General Utilities* world, with its issues of loss duplication, Section 304 anomalies, mirror progeny and the like. The corporate tax base is developing a fractal boundary even in the absence of strange loops. Dealing with strange loops will hardly simplify the picture.

These issues have been avoided in the discussion so far. Instead, the attempt has been to understand the nature of strange loops in all their complexity, and to figure out how the tax law would change if indirect self-ownership were treated as a kind of quasi-treasury stock. This does not mean, however, that the tax law must satisfy a theoretically pure ideal of equating direct and indirect self-ownership. More limited goals suffice: preventing corporations from affirmatively using strange loops to avoid tax, while avoiding an undue burden on those occasions when strange loops perform a legitimate business function.

Some changes are clearly in order. Strange loops should not make it possible for corporations to dispose of interests in appreciated corporate assets without paying corporate tax. Conversely, corporations should not be entitled to claim loss deductions on sales of what properly can be regarded as quasi-treasury stock. If a simple but overbroad rule can deal with these problems, it may be preferable to a finely crafted one that is theoretically more satisfying. These judgments, however, cannot be made without a full understanding of how strange loops work.

Appendix I
Some Theorems About Strange Loops

Theorem 1: The total value of the shares of corporations in $\{C\}$ held by individuals is equal to the asset value of the corporations:

$$\sum_j \left(\sum_k M(I_k, C_j) V(C_j) \right) = \sum_j A(C_j).$$

Proof: Condition 1 (which requires that the total outstanding shareholding be 100%) can be restated as follows:

(52)
$$1 - \sum_{i \neq j} S(C_i, C_j) = \sum_k M(I_k, C_j).$$

The equations in (8) can be added together to get:

(53)
$$\left(1 - \sum_{i \neq 1} S(C_i, C_1) \right) V(C_1) + \left(1 - \sum_{i \neq 2} S(C_i, C_2) \right) V(C_2) + \dots$$
$$+ \left(1 - \sum_{i \neq n} S(C_i, C_n) \right) V(C_n) = \sum_i A(C_i).$$

The coefficient of each term in Equation (53) is simply the left half of Equation (52), and can be replaced by the right half.

(54)
$$\sum_k M(I_k, C_1) V(C_1) + \sum_k M(I_k, C_2) V(C_2) + \dots$$
$$+ \sum_k M(I_k, C_n) V(C_n) = \sum_i A(C_i).$$

By expressing the left half of Equation (54) as a double summation and changing the arbitrary index on the right side from i to j, the result is:

(55)
$$\sum_k \left(\sum_j M(I_k, C_j) V(C_j) \right) = \sum_j A(C_j).$$

which is the result to be proved. $\therefore$

Theorem 2: The equations in (8) have a unique solution if and only if Condition 2 is true.

Proof: The proof relies on the more general principle that a set of n simultaneous equations in n unknowns has a unique solution if and only if no equation can be expressed as a weighted sum of two or

more of the others. The proof has two parts. The first part establishes that if the equations in (8) have a unique solution, then Condition 2 must be true. The second part establishes the reverse: If Condition 2 is true, then the equations must have a unique solution.

First part. Suppose Condition 2 were not true. Then there is a subset $\{D\}$ of $\{C\}$ (which may be the full set), of which for each member C_j of $\{D\}$, $S(C_i,C_j) = 0$ for every C_i outside $\{D\}$, and $M(I_k,C_j) = 0$ for every I_k. Condition 1 then implies, for each such C_j,

$$(56) \qquad \sum_{\substack{i \neq j}}^{\{D\}} S(C_i,C_j) = 1 .$$

When the summation is restricted to corporations C_i in $\{D\}$, the equations in (7) corresponding to each C_i in the subset can be summed as follows:

$$(57) \qquad \sum_{i}^{\{D\}} V(C_i) - \sum_{i}^{\{D\}} \left(\sum_{j \neq i}^{\{C\}} S(C_i,C_j) V(C_j) \right) = \sum_{i}^{\{D\}} A(C_i) .$$

The inner summation in the double summation covers the entire set $\{C\}$, and can be divided into the part that covers the subset $\{D\}$ and the part that covers its (possibly empty) complement $\{D'\}$ in $\{C\}$. Putting the latter part on the right hand side of the equation gives:

$$(58) \qquad \begin{aligned} \sum_{i}^{\{D\}} V(C_i) - \sum_{i}^{\{D\}} \left(\sum_{j \neq i}^{\{D\}} S(C_i,C_j) V(C_j) \right) = \\ \sum_{i}^{\{D\}} A(C_i) + \sum_{i}^{\{D\}} \left(\sum_{j}^{\{D'\}} S(C_i,C_j) V(C_j) \right) \end{aligned}$$

The double summation on the left half of Equation (58) can be rearranged as

$$\sum_{j}^{\{D\}} V(C_j) \sum_{\substack{i \neq j}}^{\{D\}} S(C_i C_j) .$$

which, because of Equation (56), is simply:

$$\sum_{j}^{\{D\}} V(C_j).$$

This term can be substituted for the double summation in Equation (58):

(59) $\displaystyle\sum_{i}^{\{D\}} V(C_i) - \sum_{j}^{\{D\}} V(C_j) = \sum_{i}^{\{D\}} A(C_i) + \sum_{i}^{\{D\}}\left(\sum_{j}^{\{D\}} S(C_i,C_j)V(C_j)\right).$

Since the two summations on the left half of Equation (59) cover the same elements, they cancel out to zero:

(60) $\displaystyle 0 = \sum_{i}^{\{D\}} A(C_i) + \sum_{i}^{\{D\}}\left(\sum_{j}^{\{D'\}} S(C_i,C_j)V(C_j)\right).$

Each term in the right half of (60) is defined to be greater than or equal to zero for all i, so the summation in (60) can be true only if $A(C_i) = 0$ for each C_i in $\{D\}$ and $S(C_i,C_j) = 0$ for each C_i in $\{D'\}$ and each C_j in $\{D\}$.

So far, Equation (60) shows that if a group of corporations is collectively self-owned, then a solution exists only if the asset value of each corporation is zero and the corporations in the group have no shareholding in corporations outside the group. Yet even in this case, the solution is not unique. This can be seen by observing that the equations taken into account in (57) add up to zero. This means that any equation in the group can be expressed as the sum of -1 times each of the others. Accordingly, the equations as a whole lack a unique solution.

Second part. Now suppose the equations in (8) lack a unique solution. Then some weighted sum of the equations adds up to zero. Let $\{D\}$ be the set of C_p corresponding to these equations. Let x_j be the coefficient of the jth equation in this weighted sum. Then, for each j,

(61) $\displaystyle x_j - \sum_{i \ne j}^{\{D\}} x_i S(C_i,C_j) = 0.$

Let X be the largest element of the set $\{x_j\}$, and let $y_i = x_j = X$. Then, for each j:

(62)
$$y_j - \sum_{i \neq j}^{\{D\}} y_i S(C_i, C_j) = 0 \,.$$

For at least one j, $y_i = 1$. In that case,

(63)
$$1 - \sum_{i \neq j}^{\{D\}} y_i S(C_i, C_j) = 0 \,.$$

From Condition 1, for this (and every) j,

(64)
$$\sum_{i \neq j}^{\{D\}} S(C_i, C_j) \leq 1 \,.$$

But the summation in Equation (63) contains the same terms, weighted by coefficients y_i, each of which is less than or equal to 1. Equation (63) therefore can be true only if, for that j,

(65)
$$\sum_{i \neq j}^{\{D\}} S(C_i, C_j) = 1 \,,$$

and $y_i = 1$ for all i. If 1 is then substituted for each y_i in Equation (62), then Equation (65) holds for all j. Condition 1 then implies that $M(I_k, C_j) = 0$ for all individuals I_k and all corporations C_j in $\{D\}$, and that $S(C_i, C_j) = 0$ for all corporations C_i in $\{D'\}$ and all corporations C_j in $\{D\}$. Condition 2 must therefore be false. $\therefore$

Theorem 3: The sum of the percentage interests of all individuals in each corporation is 100%:

$$\text{For all } i, \ \sum_k P(I_k, C_i) = 1 \,.$$

Proof: Since the flow matrix F is defined as $(I - S)^{-1}$, from the definition of a matrix inverse, $(I - S)F = I$, or applying the right distributive law, $F - SF = I$. Using the definition of matrix multiplication and the fact that the identity matrix has ones down its main diagonal and otherwise zeroes, the following is true for each i and j:

(66) $F(C_j, C_i) - \sum_h S(C_j, C_h) F(C_h, C_i) = 1$, if $i = j$, otherwise 0.

Summing the equations in (66) for a particular i yields:

(67) $$\sum_j F(C_j, C_i) - \sum_j \sum_h S(C_j, C_h) F(C_h, C_i) = 1.$$

Since the indices j and h in the double summation range over the same numbers, they can be substituted for each other in the double summation:

(68) $$\sum_j F(C_j, C_i) - \sum_h \sum_j S(C_h, C_j) F(C_j, C_i) = 1.$$

The two summation signs of the double summation can be reordered:

(69) $$\sum_j F(C_j, C_i) - \sum_j \sum_h S(C_h, C_j) F(C_j, C_i) = 1.$$

$F(C_j, C_i)$ does not depend on h, so it can be factored out of the inner summation:

(70) $$\sum_j F(C_j, C_i) - \sum_j F(C_j, C_i) \sum_h S(C_h, C_j) = 1.$$

The first summation is factored out of both terms:

(71) $$\sum_j F(C_j, C_i)\left(1 - \sum_k S(C_h, C_j)\right) = 1.$$

Because of Condition 1, the quantity in brackets is equal to the total individual shareholding in corporation C_j:

(72) $$\sum_j F(C_j, C_i) \sum_k M(I_k, C_j) = 1.$$

Rearrangement of summation signs yields:

(73) $$\sum_k \sum_j M(I_k, C_j) F(C_j, C_i) = 1.$$

The inner summation is equal to $P(I_k, C_i)$, based on the definition of (I_k, C_i) and the rules for matrix multiplication:

(74) $$\sum_k P(I_k, C_i) = 1,$$

which is the result to be proved. $\therefore$

Theorem 4: The outside interests of all corporations in any given corporation add up to 100%:

$$\sum_i Q(C_i, C_j) = 1, \text{ for all } j.$$

Proof: This theorem is a corollary of Theorem 3. The definition of $Q(C_i, C_j)$ provides:

(75)
$$\sum_i Q(C_i, C_j) = \sum_i F(C_i, C_j) \sum_k M(I_k, C_i).$$

The summation for all corporations C_i is:

(76)
$$Q(C_i, C_j) = F(C_i, C_j) \sum_k M(I_k, C_i).$$

The right half of Equation (76) is the same as the left half of Equation (72) except for a relettering of indices, and can be replaced by the right half of Equation (72):

$$\sum_i Q(C_i, C_j) = 1,$$

which is the result to be proved. ∴

Appendix II
An Abecedarian Guide to the Formal Theory

A   ASSET VALUE   vector on $\{C\}$; the value of underlying assets (that is, assets other than corporate stock)

C   CORPORATION   element of $\{C\}$; an entity that can issue and hold ownership interests

D   DISTRIBUTED AMOUNT   vector on $\{C\}$; the income of each corporation available for distribution

F   FLOW   matrix on $\{C\} \times \{C\}$; amount of income flowing through a corporation per unit of income originating in another, assuming full distribution

G   AFFILIATED GROUP   subset of $\{C\}$; a group of corporations joined by pairwise linkage

I   INDIVIDUAL   element of $\{I\}$; a person who holds, but does not issue, ownership interests (in other contexts, I represents the identity matrix)

L   LINKAGE   relation on $\{C\} \times \{C\}$; true of a pair of corporations if the flow from either to the other exceeds 80%

M   SHAREHOLDING   matrix on $\{I\} \times \{C\}$; percentage of direct stock ownership of an individual in a corporation

P   PERCENTAGE INTEREST   matrix on $\{I\} \times \{C\}$; percentage of direct and indirect stock ownership of an individual in a corporation

Q   OUTSIDE INTEREST   matrix on $\{C\} \times \{C\}$; flow from one corporation to another that is distributable

		to the receiving corporation's individual direct shareholders
R	RECEIVED AMOUNT	vector on $\{I\}$; total amount distributed to each individual
S	SHAREHOLDING	matrix on $\{C\} \times \{C\}$; percentage of direct stock ownership of one corporation in another
T	TAX RATE	matrix on $\{C\} \times \{C\}$; rate of tax on dividends from one corporation to another
V	VALUE	vector on $\{C\}$; the value of total assets (including stock of other corporations)
Z	SELF-OWNED INTEREST	vector on $\{C\}$; the percentage of outstanding stock that is indirectly self-owned